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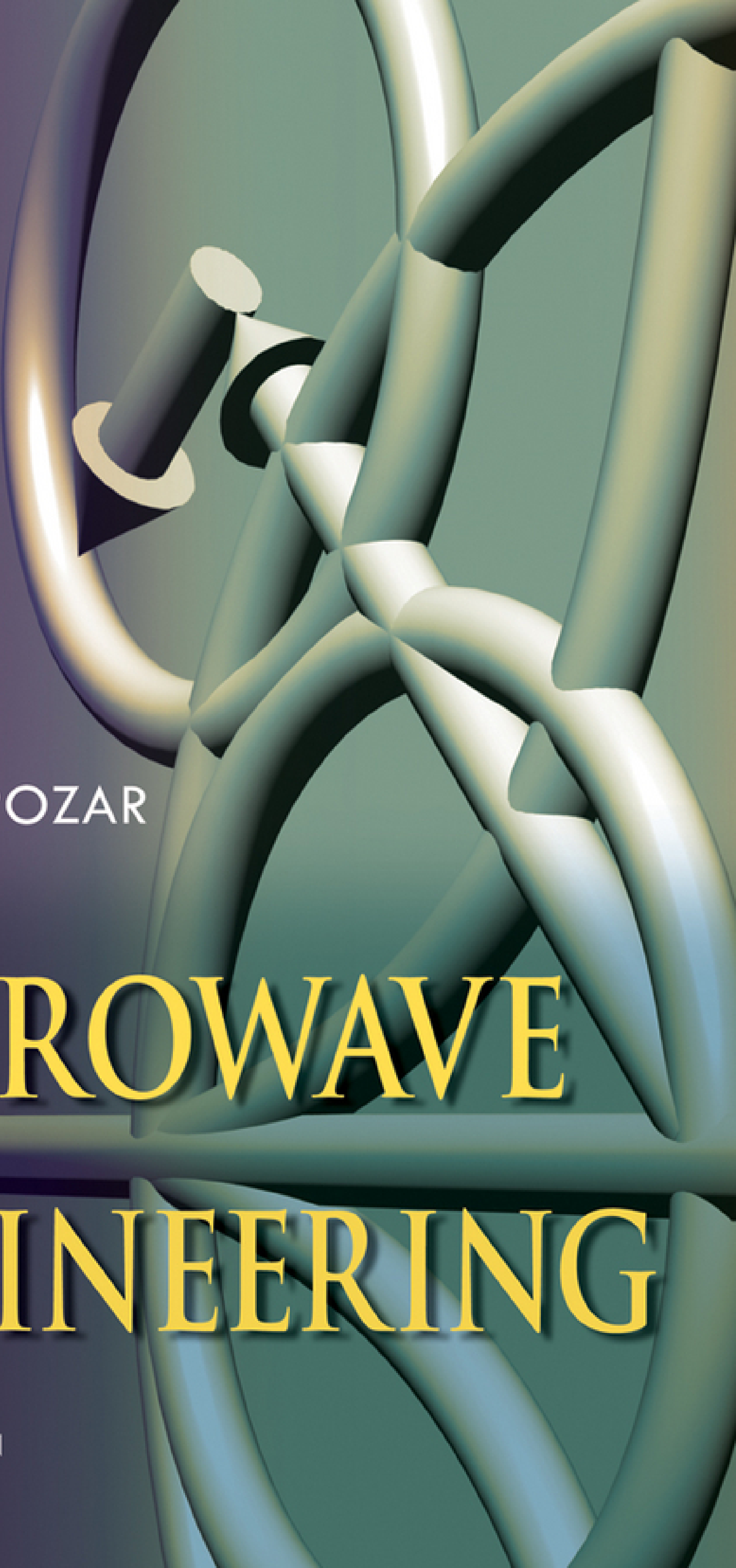
MICROWAVE ENGINEERING



FOURTH EDITION

C h a p t e r   T w o

Transmission Line Theory

Transmission line theory bridges the gap between field analysis and basic circuit theory and therefore is of significant importance in the analysis of microwave circuits and devices. As we will see, the phenomenon of wave propagation on transmission lines can be approached from an extension of circuit theory or from a specialization of Maxwell's equations; we shall present both viewpoints and show how this wave propagation is described by equations very similar to those used in Chapter 1 for plane wave propagation.

2.1 THE LUMPED-ELEMENT CIRCUIT MODEL FOR A TRANSMISSION LINE

The key difference between circuit theory and transmission line theory is electrical size. Circuit analysis assumes that the physical dimensions of the network are much smaller than the electrical wavelength, while transmission lines may be a considerable fraction of a wavelength, or many wavelengths, in size. Thus a transmission line is a *distributed-parameter* network, where voltages and currents can vary in magnitude and phase over its length, while ordinary circuit analysis deals with *lumped elements,* where voltage and current do not vary appreciably over the physical dimension of the elements.

As shown in Figure 2.1a, a transmission line is often schematically represented as a two-wire line since transmission lines (for transverse electromagnetic [TEM] wave propagation) always have at least two conductors. The piece of line of infinitesimal length Δz of Figure 2.1a can be modeled as a lumped-element circuit, as shown in Figure 2.1b, where R, L, G, and C are per-unit-length quantities defined as follows:

R = series resistance per unit length, for both conductors, in Ω/m.

L = series inductance per unit length, for both conductors, in H/m.

G = shunt conductance per unit length, in S/m.

C = shunt capacitance per unit length, in F/m.

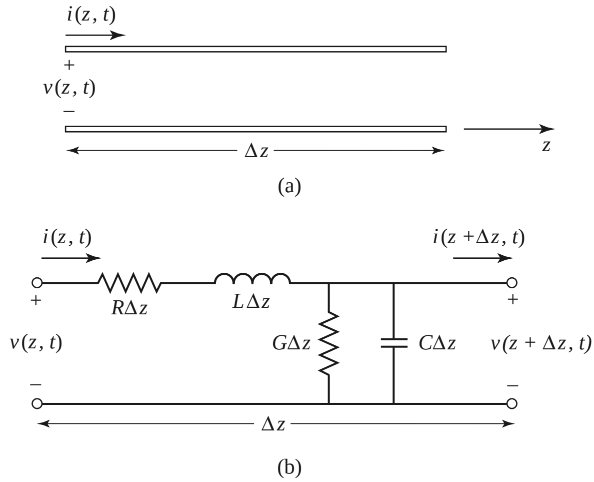


FIGURE 2.1 Voltage and current definitions and equivalent circuit for an incremental length of transmission line. (a) Voltage and current definitions. (b) Lumped-element equivalent circuit.

The series inductance L represents the total self-inductance of the two conductors, and the shunt capacitance C is due to the close proximity of the two conductors. The series resistance R represents the resistance due to the finite conductivity of the individual conductors, and the shunt conductance G is due to dielectric loss in the material between the conductors. R and G, therefore, represent loss. A finite length of transmission line can be viewed as a cascade of sections of the form shown in Figure 2.1b.

From the circuit of Figure 2.1b, Kirchhoff's voltage law can be applied to give

$$v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z, t) = 0, \tag{2.1a}$$

and Kirchhoff's current law leads to

$$i(z,t) - G\Delta z v(z+\Delta z, t) - C\Delta z \frac{\partial v(z+\Delta z, t)}{\partial t} - i(z+\Delta z, t) = 0. \tag{2.1b}$$

Dividing (2.1a) and (2.1b) by Δz and taking the limit as $\Delta z \to 0$ gives the following differential equations:

$$\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}, \tag{2.2a}$$

$$\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t}. \tag{2.2b}$$

These are the time domain form of the transmission line equations, also known as the *telegrapher equations*.

For the sinusoidal steady-state condition, with cosine-based phasors, (2.2a) and (2.2b) simplify to

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z), \tag{2.3a}$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z). \tag{2.3b}$$

Note the similarity in the form of (2.3a) and (2.3b) and Maxwell's curl equations of (1.41a) and (1.41b).

Wave Propagation on a Transmission Line

The two equations (2.3a) and (2.3b) can be solved simultaneously to give wave equations for $V(z)$ and $I(z)$:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0, \tag{2.4a}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0, \tag{2.4b}$$

where

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{2.5}$$

is the complex propagation constant, which is a function of frequency. Traveling wave solutions to (2.4) can be found as

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}, \tag{2.6a}$$

$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z}, \tag{2.6b}$$

where the $e^{-\gamma z}$ term represents wave propagation in the $+z$ direction, and the $e^{\gamma z}$ term represents wave propagation in the $-z$ direction. Applying (2.3a) to the voltage of (2.6a) gives the current on the line:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}\right).$$

Comparison with (2.6b) shows that a *characteristic impedance*, Z_0, can be defined as

$$Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \tag{2.7}$$

to relate the voltage and current on the line as follows:

$$\frac{V_o^+}{I_o^+} = Z_0 = \frac{-V_o^-}{I_o^-}.$$

Then (2.6b) can be rewritten in the following form:

$$I(z) = \frac{V_o^+}{Z_0} e^{-\gamma z} - \frac{V_o^-}{Z_0} e^{\gamma z}. \tag{2.8}$$

Converting back to the time domain, we can express the voltage waveform as

$$\begin{aligned} v(z,t) = {} & |V_o^+| \cos(\omega t - \beta z + \phi^+) e^{-\alpha z} \\ & + |V_o^-| \cos(\omega t + \beta z + \phi^-) e^{\alpha z}, \end{aligned} \tag{2.9}$$

where $\phi^{\pm}$ is the phase angle of the complex voltage $V_o^{\pm}$. Using arguments similar to those in Section 1.4, we find that the wavelength on the line is

$$\lambda = \frac{2\pi}{\beta}, \tag{2.10}$$

and the phase velocity is

$$v_p = \frac{\omega}{\beta} = \lambda f. \tag{2.11}$$

The Lossless Line

The above solution is for a general transmission line, including loss effects, and it was seen that the propagation constant and characteristic impedance were complex. In many practical cases, however, the loss of the line is very small and so can be neglected, resulting in a simplification of the results. Setting $R = G = 0$ in (2.5) gives the propagation constant as

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC},$$

or

$$\beta = \omega\sqrt{LC}, \tag{2.12a}$$

$$\alpha = 0. \tag{2.12b}$$

As expected for a lossless line, the attenuation constant α is zero. The characteristic impedance of (2.7) reduces to

$$Z_0 = \sqrt{\frac{L}{C}}, \tag{2.13}$$

which is now a real number. The general solutions for voltage and current on a lossless transmission line can then be written as

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}, \tag{2.14a}$$

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z}. \tag{2.14b}$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}, \tag{2.15}$$

and the phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}. \tag{2.16}$$

2.2 FIELD ANALYSIS OF TRANSMISSION LINES

In this section we will rederive the time-harmonic form of the telegrapher's equations starting from Maxwell's equations. We will begin by deriving the transmission line parameters (R, L, G, C) in terms of the electric and magnetic fields of the transmission line and then derive the telegrapher equations using these parameters for the specific case of a coaxial line.

Transmission Line Parameters

Consider a 1 m length of a uniform transmission line with fields $\bar{E}$ and $\bar{H}$, as shown in Figure 2.2, where S is the cross-sectional surface area of the line. Let the voltage between the conductors be $V_o e^{\pm j\beta z}$ and the current be $I_o e^{\pm j\beta z}$. The time-average stored magnetic

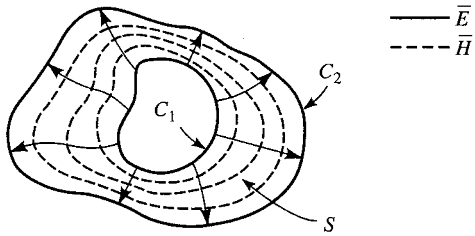


FIGURE 2.2 Field lines on an arbitrary TEM transmission line.

energy for this 1 m length of line can be written, from (1.86), as

$$W_m = \frac{\mu}{4}\int_S \bar{H}\cdot\bar{H}^* ds,$$

while circuit theory gives $W_m = L|I_o|^2/4$ in terms of the current on the line. We can thus identify the self-inductance per unit length as

$$L = \frac{\mu}{|I_o|^2}\int_S \bar{H}\cdot\bar{H}^* ds \text{ H/m}. \tag{2.17}$$

Similarly, the time-average stored electric energy per unit length can be found from (1.84) as

$$W_e = \frac{\epsilon}{4}\int_S \bar{E}\cdot\bar{E}^* ds,$$

while circuit theory gives $W_e = C|V_o|^2/4$, resulting in the following expression for the capacitance per unit length:

$$C = \frac{\epsilon}{|V_o|^2}\int_S \bar{E}\cdot\bar{E}^* ds \text{ F/m}. \tag{2.18}$$

From (1.131), the power loss per unit length due to the finite conductivity of the metallic conductors is

$$P_c = \frac{R_s}{2}\int_{C_1+C_2} \bar{H}\cdot\bar{H}^* d\ell$$

(assuming $\bar{H}$ is tangential to S), while circuit theory gives $P_c = R|I_o|^2/2$, so the series resistance R per unit length of line is

$$R = \frac{R_s}{|I_o|^2}\int_{C_1+C_2} \bar{H}\cdot\bar{H}^* dl \ \Omega\text{/m}. \tag{2.19}$$

In (2.19), $R_s = 1/\sigma\delta_s$ is the surface resistance of the conductors, and $C_1 + C_2$ represent integration paths over the conductor boundaries. From (1.92), the time-average power dissipated per unit length in a lossy dielectric is

$$P_d = \frac{\omega\epsilon''}{2}\int_S \bar{E}\cdot\bar{E}^* ds,$$

where ϵ'' is the imaginary part of the complex permittivity $\epsilon = \epsilon' - j\epsilon'' = \epsilon'(1 - j\tan\delta)$. Circuit theory gives $P_d = G|V_o|^2/2$, so the shunt conductance per unit length can be written as

$$G = \frac{\omega\epsilon''}{|V_o|^2}\int_S \bar{E}\cdot\bar{E}^* ds \text{ S/m}. \tag{2.20}$$

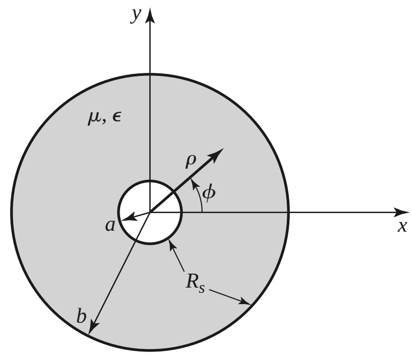
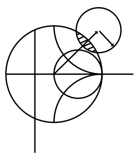


FIGURE 2.3 Geometry of a coaxial line with surface resistance R_s on the inner and outer conductors.



EXAMPLE 2.1 TRANSMISSION LINE PARAMETERS OF A COAXIAL LINE

The fields of a traveling TEM wave inside the coaxial line of Figure 2.3 can be expressed as

$$\bar{E} = \frac{V_o \hat{\rho}}{\rho \ln b/a} e^{-\gamma z},$$

$$\bar{H} = \frac{I_o \hat{\phi}}{2\pi\rho} e^{-\gamma z},$$

where γ is the propagation constant of the line. The conductors are assumed to have a surface resistivity R_s, and the material filling the space between the conductors is assumed to have a complex permittivity $\epsilon = \epsilon' - j\epsilon''$ and a permeability $\mu = \mu_0\mu_r$. Determine the transmission line parameters.

Solution

From (2.17)–(2.20) and the given fields the parameters of the coaxial line can be calculated as

$$L = \frac{\mu}{(2\pi)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \frac{1}{\rho^2} \rho d\rho d\phi = \frac{\mu}{2\pi} \ln b/a \text{ H/m},$$

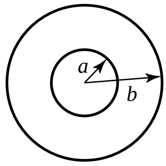
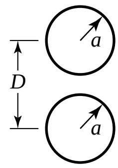
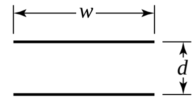
$$C = \frac{\epsilon'}{(\ln b/a)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \frac{1}{\rho^2} \rho d\rho d\phi = \frac{2\pi\epsilon'}{\ln b/a} \text{ F/m},$$

$$R = \frac{R_s}{(2\pi)^2} \left\{ \int_{\phi=0}^{2\pi} \frac{1}{a^2} a d\phi + \int_{\phi=0}^{2\pi} \frac{1}{b^2} b d\phi \right\} = \frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right) \Omega\text{/m},$$

$$G = \frac{\omega\epsilon''}{(\ln b/a)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \frac{1}{\rho^2} \rho d\rho d\phi = \frac{2\pi\omega\epsilon''}{\ln b/a} \text{ S/m.}$$ ■

Table 2.1 summarizes the parameters for coaxial, two-wire, and parallel plate lines. As we will see in the next chapter, the propagation constant, characteristic impedance, and attenuation of most transmission lines are usually derived directly from a field theory solution; the approach here of first finding the equivalent circuit parameters (L, C, R, G) is useful only for relatively simple lines. Nevertheless, it provides a helpful intuitive concept for understanding the properties of a transmission line and relates a transmission line to its equivalent circuit model.

TABLE 2.1 Transmission Line Parameters for Some Common Lines

	COAX	TWO-WIRE	PARALLEL PLATE
			
L	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$	$\frac{\mu d}{w}$
C	$\frac{2\pi\epsilon'}{\ln b/a}$	$\frac{\pi\epsilon'}{\cosh^{-1}(D/2a)}$	$\frac{\epsilon' w}{d}$
R	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{w}$
G	$\frac{2\pi\omega\epsilon''}{\ln b/a}$	$\frac{\pi\omega\epsilon''}{\cosh^{-1}(D/2a)}$	$\frac{\omega\epsilon'' w}{d}$

The Telegrapher Equations Derived from Field Analysis of a Coaxial Line

We now show that the telegrapher equations of (2.3), derived using circuit theory, can also be obtained from Maxwell's equations. We will consider the specific geometry of the coaxial line of Figure 2.3. Although we will treat TEM wave propagation more generally in the next chapter, the present discussion should provide some insight into the relationship of circuit and field quantities.

A TEM wave on the coaxial line of Figure 2.3 will be characterized by $E_z = H_z = 0$; furthermore, due to azimuthal symmetry, the fields will have no ϕ variation, so $\partial/\partial\phi = 0$. The fields inside the coaxial line will satisfy Maxwell's curl equations,

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}, \tag{2.21a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}, \tag{2.21b}$$

where $\epsilon = \epsilon' - j\epsilon''$ may be complex to allow for a lossy dielectric filling. Conductor loss will be ignored here. A rigorous field analysis of conductor loss can be carried out but at this point would tend to obscure our purpose; the interested reader is referred to references [1] and [2].

Expanding (2.21a) and (2.21b) gives the following two vector equations:

$$-\hat{\rho}\frac{\partial E_\phi}{\partial z} + \hat{\phi}\frac{\partial E_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi) = -j\omega\mu(\hat{\rho}H_\rho + \hat{\phi}H_\phi), \tag{2.22a}$$

$$-\hat{\rho}\frac{\partial H_\phi}{\partial z} + \hat{\phi}\frac{\partial H_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi) = j\omega\epsilon(\hat{\rho}E_\rho + \hat{\phi}E_\phi). \tag{2.22b}$$

Since the $\hat{z}$ components of these two equations must vanish, it is seen that E_ϕ and H_ϕ must have the forms

$$E_\phi = \frac{f(z)}{\rho}, \tag{2.23a}$$

$$H_\phi = \frac{g(z)}{\rho}. \tag{2.23b}$$

To satisfy the boundary condition that $E_\phi = 0$ at $\rho = a,\ b$, we must have $E_\phi = 0$ everywhere, due to the form of E_ϕ in (2.23a). Then from the $\hat{\rho}$ component of (2.22a), it is seen that $H_\rho = 0$. With these results, (2.22) can be reduced to

$$\frac{\partial E_\rho}{\partial z} = -j\omega\mu H_\phi, \tag{2.24a}$$

$$\frac{\partial H_\phi}{\partial z} = -j\omega\epsilon E_\rho. \tag{2.24b}$$

From the form of H_ϕ in (2.23b) and (2.24a), E_ρ must be of the form

$$E_\rho = \frac{h(z)}{\rho}. \tag{2.25}$$

Using (2.23b) and (2.25) in (2.24) gives

$$\frac{\partial h(z)}{\partial z} = -j\omega\mu g(z), \tag{2.26a}$$

$$\frac{\partial g(z)}{\partial z} = -j\omega\epsilon h(z). \tag{2.26b}$$

The voltage between the two conductors can be evaluated as

$$V(z) = \int_{\rho=a}^{b} E_\rho(\rho, z) d\rho = h(z) \int_{\rho=a}^{b} \frac{d\rho}{\rho} = h(z) \ln \frac{b}{a}, \tag{2.27a}$$

and the total current on the inner conductor at $\rho = a$ can be evaluated using (2.23b) as

$$I(z) = \int_{\phi=0}^{2\pi} H_\phi(a, z) a d\phi = 2\pi g(z). \tag{2.27b}$$

Then $h(z)$ and $g(z)$ can be eliminated from (2.26) by using (2.27) to give

$$\frac{\partial V(z)}{\partial z} = -j\frac{\omega\mu \ln b/a}{2\pi} I(z),$$

$$\frac{\partial I(z)}{\partial z} = -j\omega(\epsilon' - j\epsilon'')\frac{2\pi V(z)}{\ln b/a}.$$

Finally, using the results for L, G, and C for a coaxial line as derived earlier, we obtain the telegrapher equations as

$$\frac{\partial V(z)}{\partial z} = -j\omega L I(z), \tag{2.28a}$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)V(z). \tag{2.28b}$$

This result excludes R, the series resistance, since the conductors were assumed to have perfect conductivity. A similar analysis can be carried out for other simple transmission lines.

Propagation Constant, Impedance, and Power Flow for the Lossless Coaxial Line

Equations (2.24a) and (2.24b) for E_ρ and H_ϕ can be simultaneously solved to yield a wave equation for E_ρ (or H_ϕ):

$$\frac{\partial^2 E_\rho}{\partial z^2} + \omega^2 \mu \epsilon E_\rho = 0, \tag{2.29}$$

from which it is seen that the propagation constant is $\gamma^2 = -\omega^2 \mu \epsilon$, which, for lossless media, reduces to

$$\beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{LC}, \tag{2.30}$$

where the last result is from (2.12). Observe that this propagation constant is of the same form as that for plane waves in a lossless dielectric medium. This is a general result for TEM transmission lines.

The wave impedance for the coaxial line is defined as $Z_w = E_\rho / H_\phi$, which can be calculated from (2.24a), assuming an $e^{-j\beta z}$ dependence, to give

$$Z_w = \frac{E_\rho}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\mu/\epsilon} = \eta. \tag{2.31}$$

This wave impedance is seen to be identical to the intrinsic impedance of the medium, η, and is a general result for TEM transmission lines.

The characteristic impedance of the coaxial line is defined as

$$Z_0 = \frac{V_o}{I_o} = \frac{E_\rho \ln b/a}{2\pi H_\phi} = \frac{\eta \ln b/a}{2\pi} = \sqrt{\frac{\mu}{\epsilon}}\frac{\ln b/a}{2\pi}, \tag{2.32}$$

where the forms for E_ρ and H_ϕ from Example 2.1 have been used. The characteristic impedance is geometry dependent and will be different for other transmission line configurations.

Finally, the power flow (in the z direction) on the coaxial line may be computed from the Poynting vector as

$$P = \frac{1}{2}\int_s \bar{E} \times \bar{H}^* \cdot d\bar{s} = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b} \frac{V_o I_o^*}{2\pi\rho^2 \ln b/a}\rho d\rho d\phi = \frac{1}{2}V_o I_o^*, \tag{2.33}$$

a result that is in clear agreement with circuit theory. This shows that the flow of power in a transmission line takes place entirely via the electric and magnetic fields between the two conductors; power is not transmitted through the conductors themselves. As we will see later, for the case of finite conductivity, power may enter the conductors, but this power is then lost as heat and is not delivered to the load.

2.3 THE TERMINATED LOSSLESS TRANSMISSION LINE

Figure 2.4 shows a lossless transmission line terminated in an arbitrary load impedance Z_L. This problem will illustrate wave reflection on transmission lines, a fundamental property of distributed systems.

Assume that an incident wave of the form $V_o^+ e^{-j\beta z}$ is generated from a source at $z < 0$. We have seen that the ratio of voltage to current for such a traveling wave is Z_0, the characteristic impedance of the line. However, when the line is terminated in an arbitrary load $Z_L \neq Z_0$, the ratio of voltage to current at the load must be Z_L. Thus, a reflected wave

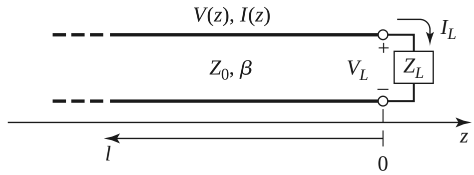


FIGURE 2.4 A transmission line terminated in a load impedance Z_L.

must be excited with the appropriate amplitude to satisfy this condition. The total voltage on the line can then be written as in (2.14a), as a sum of incident and reflected waves:

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}. \tag{2.34a}$$

Similarly, the total current on the line is described by (2.14b):

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z}. \tag{2.34b}$$

The total voltage and current at the load are related by the load impedance, so at $z = 0$ we must have

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-} Z_0.$$

Solving for V_o^- gives

$$V_o^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_o^+.$$

The amplitude of the reflected voltage wave normalized to the amplitude of the incident voltage wave is defined as the *voltage reflection coefficient*, Γ:

$$\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{2.35}$$

The total voltage and current waves on the line can then be written as

$$V(z) = V_o^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z} \right), \tag{2.36a}$$

$$I(z) = \frac{V_o^+}{Z_0} \left(e^{-j\beta z} - \Gamma e^{j\beta z} \right). \tag{2.36b}$$

From these equations it is seen that the voltage and current on the line consist of a superposition of an incident and a reflected wave; such waves are called *standing waves*. Only when $\Gamma = 0$ is there no reflected wave. To obtain $\Gamma = 0$, the load impedance Z_L must be equal to the characteristic impedance Z_0 of the transmission line, as seen from (2.35). Such a load is said to be *matched* to the line since there is no reflection of the incident wave.

Now consider the time-average power flow along the line at the point z:

$$P_{\text{avg}} = \frac{1}{2} \text{Re}\left\{ V(z) I(z)^* \right\} = \frac{1}{2} \frac{|V_o^+|^2}{Z_0} \text{Re}\left\{ 1 - \Gamma^* e^{-2j\beta z} + \Gamma e^{2j\beta z} - |\Gamma|^2 \right\},$$

where (2.36) has been used. The middle two terms in the brackets are of the form $A - A^* = 2j \ \text{Im}\{A\}$ and so are purely imaginary. This simplifies the result to

$$P_{\text{avg}} = \frac{1}{2} \frac{|V_o^+|^2}{Z_0} \left(1 - |\Gamma|^2 \right), \tag{2.37}$$

which shows that the average power flow is constant at any point on the line and that the total power delivered to the load (P_{avg}) is equal to the incident power ($|V_o^+|^2/2Z_0$) minus the reflected power ($|V_o|^2|\Gamma|^2/2Z_0$). If $\Gamma = 0$, maximum power is delivered to the load, while no power is delivered for $|\Gamma| = 1$. The above discussion assumes that the generator is matched, so that there is no re-reflection of the reflected wave from $z < 0$.

When the load is mismatched, not all of the available power from the generator is delivered to the load. This "loss" is called *return loss* (RL), and is defined (in dB) as

$$\mathrm{RL} = -20 \log |\Gamma| \text{ dB}, \tag{2.38}$$

so that a matched load ($\Gamma = 0$) has a return loss of ∞ dB (no reflected power), while a total reflection ($|\Gamma| = 1$) has a return loss of 0 dB (all incident power is reflected). Note that return loss is a nonnegative number for reflection from a passive network.

If the load is matched to the line, $\Gamma = 0$ and the magnitude of the voltage on the line is $|V(z)| = |V_o^+|$, which is a constant. Such a line is sometimes said to be *flat*. When the load is mismatched, however, the presence of a reflected wave leads to standing waves, and the magnitude of the voltage on the line is not constant. Thus, from (2.36a),

$$\begin{aligned} |V(z)| &= |V_o^+||1 + \Gamma e^{2j\beta z}| = |V_o^+||1 + \Gamma e^{-2j\beta\ell}| \\ &= |V_o^+||1 + |\Gamma| e^{j(\theta - 2\beta\ell)}|, \end{aligned} \tag{2.39}$$

where $\ell = -z$ is the positive distance measured from the load at $z = 0$, and θ is the phase of the reflection coefficient ($\Gamma = |\Gamma|e^{j\theta}$). This result shows that the voltage magnitude oscillates with position z along the line. The maximum value occurs when the phase term $e^{j(\theta - 2\beta\ell)} = 1$ and is given by

$$V_{\max} = |V_o^+|(1 + |\Gamma|). \tag{2.40a}$$

The minimum value occurs when the phase term $e^{j(\theta - 2\beta\ell)} = -1$ and is given by

$$V_{\min} = |V_o^+|(1 - |\Gamma|). \tag{2.40b}$$

As $|\Gamma|$ increases, the ratio of $V_{\max}$ to $V_{\min}$ increases, so a measure of the mismatch of a line, called the *standing wave ratio* (SWR), can be defined as

$$\mathrm{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}. \tag{2.41}$$

This quantity is also known as the *voltage standing wave ratio* and is sometimes identified as VSWR. From (2.41) it is seen that SWR is a real number such that $1 \leq \mathrm{SWR} \leq \infty$, where SWR $= 1$ implies a matched load.

From (2.39), it is seen that the distance between two successive voltage maxima (or minima) is $\ell = 2\pi/2\beta = \pi\lambda/2\pi = \lambda/2$, while the distance between a maximum and a minimum is $\ell = \pi/2\beta = \lambda/4$, where λ is the wavelength on the transmission line.

The reflection coefficient of (2.35) was defined as the ratio of the reflected to the incident voltage wave amplitudes at the load ($\ell = 0$), but this quantity can be generalized to any point ℓ along the line as follows. From (2.34a), with $z = -\ell$, the ratio of the reflected component to the incident component is

$$\Gamma(\ell) = \frac{V_o^- e^{-j\beta\ell}}{V_o^+ e^{j\beta\ell}} = \Gamma(0) e^{-2j\beta\ell}, \tag{2.42}$$

where $\Gamma(0)$ is the reflection coefficient at $z = 0$, as given by (2.35). This result is useful when transforming the effect of a load mismatch down the line.

We have seen that the real power flow on the line is a constant (for a lossless line) but that the voltage amplitude, at least for a mismatched line, is oscillatory with position on the line. The perceptive reader may therefore have concluded that the impedance seen looking into the line must vary with position, and this is indeed the case. At a distance $\ell = -z$ from the load, the input impedance seen looking toward the load is

$$Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = \frac{V_o^+\left(e^{j\beta\ell} + \Gamma e^{-j\beta\ell}\right)}{V_o^+\left(e^{j\beta\ell} - \Gamma e^{-j\beta\ell}\right)} Z_0 = \frac{1 + \Gamma e^{-2j\beta\ell}}{1 - \Gamma e^{-2j\beta\ell}} Z_0, \tag{2.43}$$

where (2.36a,b) have been used for $V(z)$ and $I(z)$. A more usable form may be obtained by using (2.35) for Γ in (2.43):

$$\begin{aligned} Z_{\text{in}} &= Z_0 \frac{(Z_L + Z_0)e^{j\beta\ell} + (Z_L - Z_0)e^{-j\beta\ell}}{(Z_L + Z_0)e^{j\beta\ell} - (Z_L - Z_0)e^{-j\beta\ell}} \\ &= Z_0 \frac{Z_L \cos\beta\ell + jZ_0 \sin\beta\ell}{Z_0 \cos\beta\ell + jZ_L \sin\beta\ell} \\ &= Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}. \end{aligned} \tag{2.44}$$

This is an important result giving the input impedance of a length of transmission line with an arbitrary load impedance. We will refer to this result as the *transmission line impedance equation;* some special cases will be considered next.

Special Cases of Lossless Terminated Lines

A number of special cases of lossless terminated transmission lines will frequently appear in our work, so it is appropriate to consider the properties of such cases here.

Consider first the transmission line circuit shown in Figure 2.5, where a line is terminated in a short circuit, $Z_L = 0$. From (2.35) it is seen that the reflection coefficient for a short circuit load is $\Gamma = -1$; it then follows from (2.41) that the standing wave ratio is infinite. From (2.36) the voltage and current on the line are

$$V(z) = V_o^+\left(e^{-j\beta z} - e^{j\beta z}\right) = -2jV_o^+ \sin\beta z, \tag{2.45a}$$

$$I(z) = \frac{V_o^+}{Z_0}\left(e^{-j\beta z} + e^{j\beta z}\right) = \frac{2V_o^+}{Z_0} \cos\beta z, \tag{2.45b}$$

which shows that $V = 0$ at the load (as expected, for a short circuit), while the current is a maximum there. From (2.44), or the ratio $V(-\ell)/I(-\ell)$, the input impedance is

$$Z_{\text{in}} = jZ_0 \tan\beta\ell, \tag{2.45c}$$

which is seen to be purely imaginary for any length ℓ and to take on all values between $+j\infty$ and $-j\infty$. For example, when $\ell = 0$ we have $Z_{\text{in}} = 0$, but for $\ell = \lambda/4$ we have $Z_{\text{in}} = \infty$ (open circuit). Equation (2.45c) also shows that the impedance is periodic in ℓ,

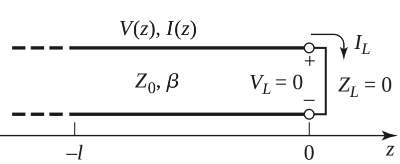


FIGURE 2.5 A transmission line terminated in a short circuit.

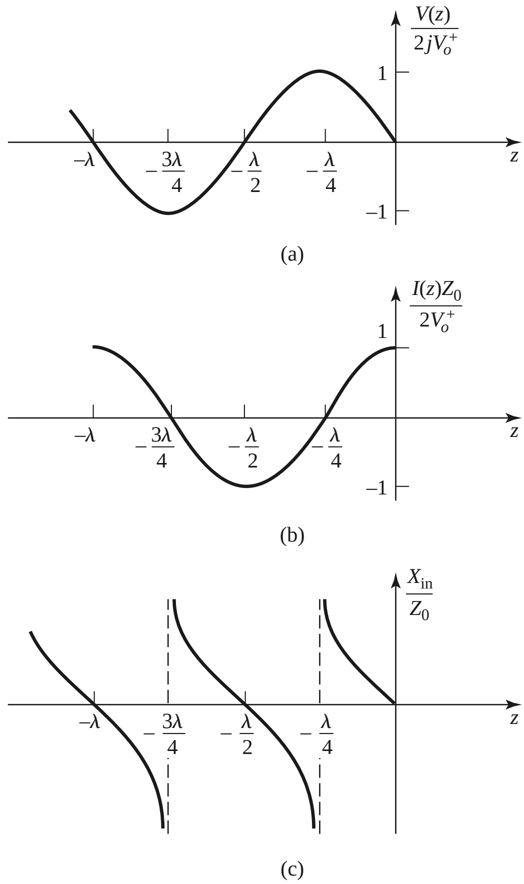


FIGURE 2.6 (a) Voltage, (b) current, and (c) impedance ($R_{\text{in}} = 0$ or ∞) variation along a short-circuited transmission line.

repeating for multiples of $\lambda/2$. The voltage, current, and input reactance for the short-circuited line are plotted in Figure 2.6.

Next consider the open-circuited line shown in Figure 2.7, where $Z_L = \infty$. Dividing the numerator and denominator of (2.35) by Z_L and allowing $Z_L \to \infty$ shows that the reflection coefficient for this case is $\Gamma = 1$, and the standing wave ratio is again infinite. From (2.36) the voltage and current on the line are

$$V(z) = V_o^+\left(e^{-j\beta z} + e^{j\beta z}\right) = 2V_o^+ \cos\beta z, \tag{2.46a}$$

$$I(z) = \frac{V_o^+}{Z_0}\left(e^{-j\beta z} - e^{j\beta z}\right) = \frac{-2jV_o^+}{Z_0} \sin\beta z, \tag{2.46b}$$

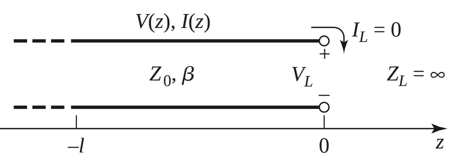
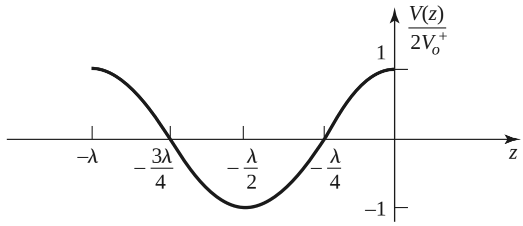
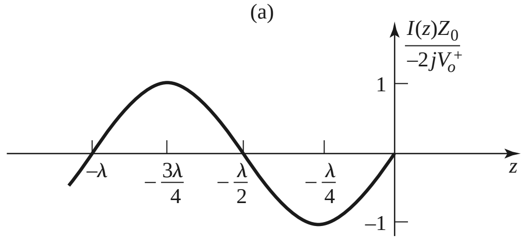


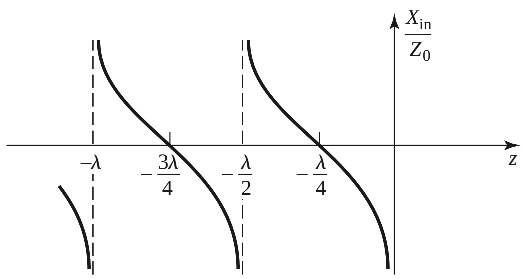
FIGURE 2.7 A transmission line terminated in an open circuit.



(a)



(b)



(c)

FIGURE 2.8 (a) Voltage, (b) current, and (c) impedance ($R_{in} = 0$ or ∞) variation along an open-circuited transmission line.

which shows that now $I = 0$ at the load, as expected for an open circuit, while the voltage is a maximum. The input impedance is

$$Z_{in} = -jZ_0 \cot \beta\ell, \tag{2.46c}$$

which is also purely imaginary for any length, ℓ. The voltage, current, and input reactance of the open-circuited line are plotted in Figure 2.8.

Now consider terminated transmission lines with some special lengths. If $\ell = \lambda/2$, (2.44) shows that

$$Z_{in} = Z_L, \tag{2.47}$$

meaning that a half-wavelength line (or any multiple of $\lambda/2$) does not alter or transform the load impedance, regardless of its characteristic impedance.

If the line is a quarter-wavelength long or, more generally, $\ell = \lambda/4 + n\lambda/2$, for $n = 1, 2, 3, \ldots$, (2.44) shows that the input impedance is given by

$$Z_{in} = \frac{Z_0^2}{Z_L}. \tag{2.48}$$

Such a line is known as a *quarter-wave transformer* because it has the effect of transforming the load impedance in an inverse manner, depending on the characteristic impedance of the line. We will study this case more thoroughly in Section 2.5.

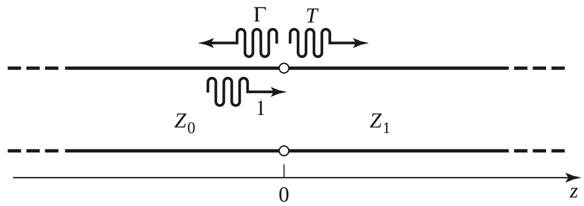


FIGURE 2.9 Reflection and transmission at the junction of two transmission lines with different characteristic impedances.

Next consider a transmission line of characteristic impedance Z_0 feeding a line of different characteristic impedance, Z_1, as shown in Figure 2.9. If the load line is infinitely long, or if it is terminated in its own characteristic impedance, so that there are no reflections from its far end, then the input impedance seen by the feed line is Z_1, so that the reflection coefficient Γ is

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}. \tag{2.49}$$

Not all of the incident wave is reflected; some is transmitted onto the second line with a voltage amplitude given by a transmission coefficient.

From (2.36a) the voltage for $z < 0$ is

$$V(z) = V_o^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right), \quad z < 0, \tag{2.50a}$$

where V_o^+ is the amplitude of the incident voltage wave on the feed line. The voltage wave for $z > 0$, in the absence of reflections, is outgoing only and can be written as

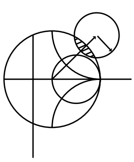
$$V(z) = V_o^+ T e^{-j\beta z} \quad \text{for } z > 0. \tag{2.50b}$$

Equating these voltages at $z = 0$ gives the *transmission coefficient*, T, as

$$T = 1 + \Gamma = 1 + \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{2Z_1}{Z_1 + Z_0}. \tag{2.51}$$

The transmission coefficient between two points in a circuit is often expressed in dB as the *insertion loss*, IL,

$$\text{IL} = -20 \log |T| \text{dB}. \tag{2.52}$$



POINT OF INTEREST: Decibels and Nepers

Often the ratio of two power levels P_1 and P_2 in a microwave system is expressed in decibels (dB) as

$$10 \log \frac{P_1}{P_2} \text{ dB}.$$

Thus, a power ratio of 2 is equivalent to 3 dB, while a power ratio of 0.1 is equivalent to -10 dB. Using power ratios in dB makes it easy to calculate power loss or gain through a series of components since multiplicative loss or gain factors can be accounted for by adding the loss or gain in dB for each stage. For example, a signal passing through a 6 dB attenuator followed by a 23 dB amplifier will have an overall gain of $23 - 6 = 17$ dB.

Decibels are used only to represent power ratios, but if $P_1 = V_1^2/R_1$ and $P_2 = V_2^2/R_2$, then the resulting power ratio in terms of voltage ratios is

$$10 \log \frac{V_1^2 R_2}{V_2^2 R_1} = 20 \log \frac{V_1}{V_2} \sqrt{\frac{R_2}{R_1}} \text{ dB},$$

where $R_1,\ R_2$ are the load resistances and $V_1,\ V_2$ are the voltages appearing across these loads. If the load resistances are equal, then this formula simplifies to

$$20 \log \frac{V_1}{V_2} \text{ dB}.$$

The ratio of voltages across equal load resistances can also be expressed in terms of nepers (Np) as

$$\ln \frac{V_1}{V_2} \text{ Np}.$$

The corresponding expression in terms of powers is

$$\frac{1}{2} \ln \frac{P_1}{P_2} \text{ Np},$$

since voltage is proportional to the square root of power. Transmission line attenuation is sometimes expressed in nepers. Since 1 Np corresponds to a power ratio of e^2, the conversion between nepers and decibels is

$$1 \text{ Np} = 10 \log e^2 = 8.686 \text{ dB}.$$

Absolute power can also be expressed in decibel notation if a reference power level is assumed. If we let $P_2 = 1$ mW, then the power P_1 can be expressed in dBm as

$$10 \log \frac{P_1}{1 \text{ mW}} \text{ dBm}$$

Thus a power of 1 mW is equivalent to 0 dBm, while a power of 1 W is equivalent to 30 dBm, and so on.

2.4 THE SMITH CHART

The Smith chart, shown in Figure 2.10, is a graphical aid that can be very useful for solving transmission line problems. Although there are a number of other impedance and reflection coefficient charts that can be used for such problems [3], the Smith chart is probably the best known and most widely used. It was developed in 1939 by P. Smith at the Bell Telephone Laboratories [4]. The reader might feel that, in this day of personal computers and computer-aided design (CAD) tools, graphical solutions have no place in modern engineering. The Smith chart, however, is more than just a graphical technique. Besides being an integral part of much of the current CAD software and test equipment for microwave design, the Smith chart provides a useful way of visualizing transmission line phenomenon without the need for detailed numerical calculations. A microwave engineer can develop a good intuition about transmission line and impedance-matching problems by learning to think in terms of the Smith chart.

At first glance the Smith chart may seem intimidating, but the key to its understanding is to realize that it is based on a polar plot of the voltage reflection coefficient, Γ. Let the reflection coefficient be expressed in magnitude and phase (polar) form as $\Gamma = |\Gamma|e^{j\theta}$. Then the magnitude $|\Gamma|$ is plotted as a radius ($|\Gamma| \leq 1$) from the center of the chart, and the angle θ ($-180° \leq \theta \leq 180°$) is measured counterclockwise from the right-hand side of

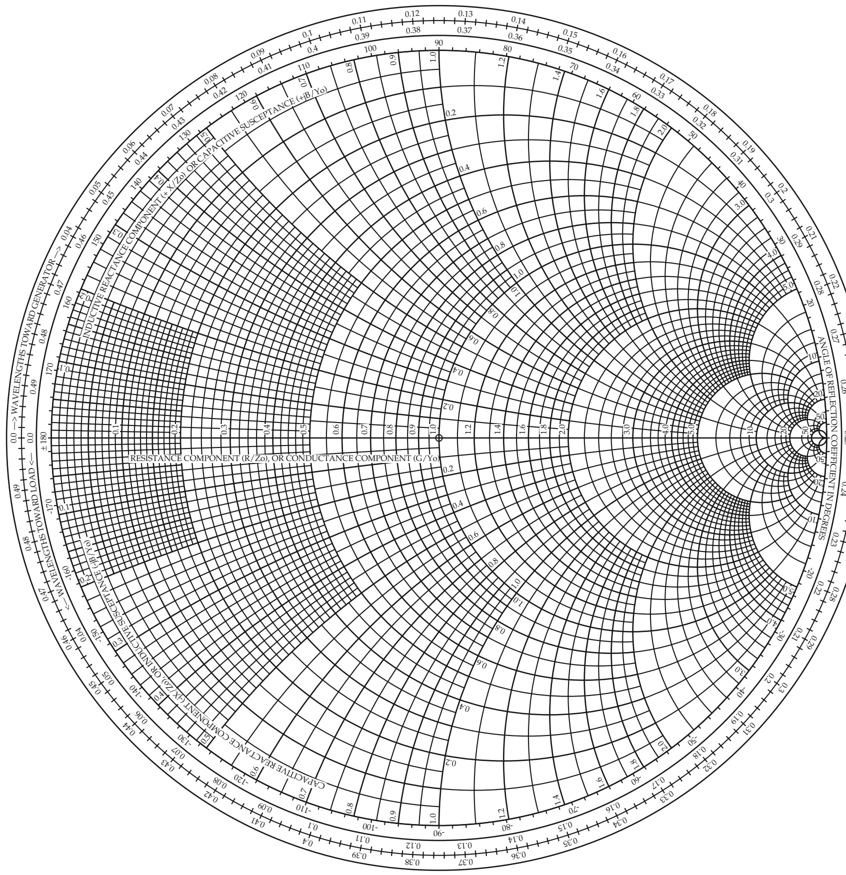


FIGURE 2.10 The Smith chart.

the horizontal diameter. Any passively realizable ($|\Gamma| \leq 1$) reflection coefficient can then be plotted as a unique point on the Smith chart.

The real utility of the Smith chart, however, lies in the fact that it can be used to convert from reflection coefficients to normalized impedances (or admittances) and vice versa by using the impedance (or admittance) circles printed on the chart. When dealing with impedances on a Smith chart, normalized quantities are generally used, which we will denote by lowercase letters. The normalization constant is usually the characteristic impedance of the transmission line. Thus, $z = Z/Z_0$ represents the normalized version of the impedance Z.

If a lossless line of characteristic impedance Z_0 is terminated with a load impedance Z_L, the reflection coefficient at the load can be written from (2.35) as

$$\Gamma = \frac{z_L - 1}{z_L + 1} = |\Gamma| e^{j\theta}, \tag{2.53}$$

where $z_L = Z_L/Z_0$ is the normalized load impedance. This relation can be solved for z_L in terms of Γ to give [or, from (2.43) with $\ell = 0$]

$$z_L = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}}. \tag{2.54}$$

This complex equation can be reduced to two real equations by writing Γ and z_L in terms of their real and imaginary parts, $\Gamma = \Gamma_r + j\Gamma_i$, and $z_L = r_L + jx_L$, giving

$$r_L + jx_L = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}.$$

The real and imaginary parts of this equation can be separated by multiplying the numerator and denominator by the complex conjugate of the denominator to give

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}, \tag{2.55a}$$

$$x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}. \tag{2.55b}$$

Rearranging (2.55) gives

$$\left(\Gamma_r - \frac{r_L}{1 + r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + r_L}\right)^2, \tag{2.56a}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2, \tag{2.56b}$$

which are seen to represent two families of circles in the Γ_r, Γ_i plane. Resistance circles are defined by (2.56a) and reactance circles are defined by (2.56b). For example, the $r_L = 1$ circle has its center at $\Gamma_r = 0.5,\ \Gamma_i = 0$, and has a radius of 0.5, and so it passes through the center of the Smith chart. All of the resistance circles of (2.56a) have centers on the horizontal $\Gamma_i = 0$ axis and pass through the $\Gamma = 1$ point on the right-hand side of the chart. The centers of all of the reactance circles of (2.56b) lie on the vertical $\Gamma_r = 1$ line (off the chart), and these circles also pass through the $\Gamma = 1$ point. The resistance and reactance circles are orthogonal.

The Smith chart can also be used to graphically solve the transmission line impedance equation of (2.44) since this can be written in terms of the generalized reflection coefficient as

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma e^{-2j\beta\ell}}{1 - \Gamma e^{-2j\beta\ell}}, \tag{2.57}$$

where Γ is the reflection coefficient at the load and ℓ is the (positive) length of transmission line. We then see that (2.57) is of the same form as (2.54), differing only by the phase angles of the Γ terms. Thus, if we have plotted the reflection coefficient $|\Gamma|e^{j\theta}$ at the load, the normalized input impedance seen looking into a length ℓ of transmission line terminated with z_L can be found by rotating the point clockwise by an amount $2\beta\ell$ (subtracting $2\beta\ell$ from θ) around the center of the chart. The radius stays the same since the magnitude of Γ does not change with position along the line (assuming a lossless line).

To facilitate such rotations, the Smith chart has scales around its periphery calibrated in electrical wavelengths, toward and away from the "generator" (which simply means the direction away from the load). These scales are relative, so only the difference in wavelengths between two points on the Smith chart is meaningful. The scales cover a range of 0 to 0.5 wavelength, which reflects the fact that the Smith chart automatically includes the periodicity of transmission line phenomenon. Thus, a line of length $\lambda/2$ (or any multiple) requires a rotation of $2\beta\ell = 2\pi$ around the center of the chart, bringing the point back to its original position, showing that the input impedance of a load seen through a $\lambda/2$ line is unchanged.

We will now illustrate the use of the Smith chart for a variety of typical transmission line problems through examples.

EXAMPLE 2.2 BASIC SMITH CHART OPERATIONS

A load impedance of $40 + j70\ \Omega$ terminates a $100\ \Omega$ transmission line that is 0.3λ long. Find the reflection coefficient at the load, the reflection coefficient at the input to the line, the input impedance, the standing wave ratio on the line, and the return loss.

Solution

The normalized load impedance is

$$z_L = \frac{Z_L}{Z_0} = 0.4 + j0.7,$$

which can be plotted on the Smith chart as shown in Figure 2.11. By using a drawing compass and the voltage coefficient scale printed below the chart, one can read off the reflection coefficient magnitude at the load as $|\Gamma| = 0.59$. This same compass setting can then be applied to the standing wave ratio (SWR) scale to read SWR $= 3.87$ and to the return loss (RL) (in dB) scale to read RL $=$ 4.6 dB.

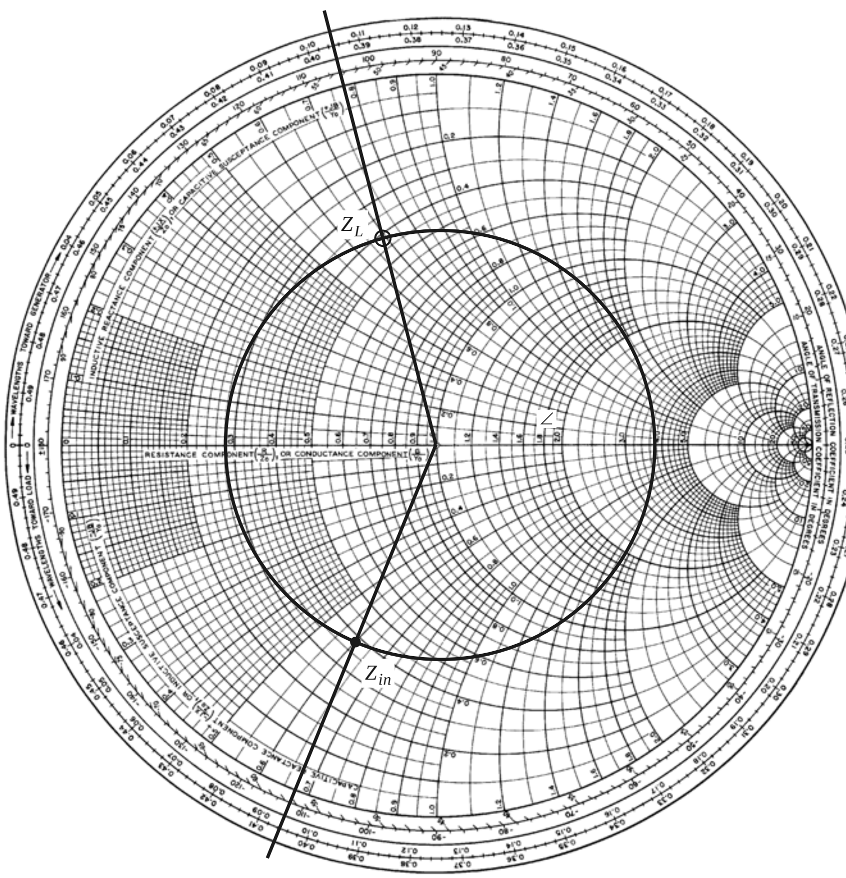


FIGURE 2.11 Smith chart for Example 2.2.

Now draw a radial line through the load impedance point and read the angle of the reflection coefficient at the load from the outer scale of the chart as 104°.

Now draw an SWR circle through the load impedance point. Reading the reference position of the load on the wavelengths-toward-generator (WTG) scale gives a value of 0.106λ. Moving down the line 0.3λ toward the generator brings us to 0.406λ on the WTG scale. Drawing a radial line at this position gives the normalized input impedance at the intersection with SWR circle of $z_{\text{in}} = 0.365 - j0.611$. Then the input impedance of the line is

$$Z_{\text{in}} = Z_0 z_{\text{in}} = 36.5 - j61.1\ \Omega.$$

The reflection coefficient at the input still has a magnitude of $|\Gamma| = 0.59$; the phase is read from the radial line at the phase scale as 248°. ■

The Combined Impedance–Admittance Smith Chart

The Smith chart can be used for normalized admittance in the same way that it is used for normalized impedances, and it can be used to convert between impedance and admittance. The latter technique is based on the fact that, in normalized form, the input impedance of a load z_L connected to a $\lambda/4$ line is, from (2.44),

$$z_{\text{in}} = 1/z_L,$$

which has the effect of converting a normalized impedance to a normalized admittance.

Since a complete revolution around the Smith chart corresponds to a line length of $\lambda/2$, a $\lambda/4$ transformation is equivalent to a 180° rotation; this is also equivalent to imaging a given impedance (or admittance) point across the center of the chart to obtain the corresponding admittance (or impedance) point.

Thus, a Smith chart can be used for both impedance and admittance calculations during the solution of a given problem. At different stages of the solution, then, the chart may be either an *impedance Smith chart* or an *admittance Smith chart*. This procedure can be made less confusing by using a Smith chart that has a superposition of the scales for a regular Smith chart and the scales of a Smith chart that has been rotated by180°, as shown in Figure 2.12. Such a chart is referred to as an *impedance and admittance Smith chart* and usually has different-colored scales for impedance and admittance.

EXAMPLE 2.3 SMITH CHART OPERATIONS USING ADMITTANCES

A load of $Z_L = 100 + j50\ \Omega$ terminates a 50 Ω line. What are the load admittance and input admittance if the line is 0.15λ long?

Solution

The normalized load impedance is $z_L = 2 + j1$. A standard Smith chart can be used for this problem by initially considering it as an impedance chart and plotting z_L and the SWR circle. Conversion to admittance can be accomplished with a $\lambda/4$ rotation of z_L (easily obtained by drawing a straight line through z_L and the center of the chart to intersect the other side of the SWR circle). The chart can now be considered as an admittance chart, and the input admittance can be found by rotating 0.15λ from y_L.

Alternatively, we can use the combined zy chart of Figure 2.12, where conversion between impedance and admittance is accomplished merely by reading the

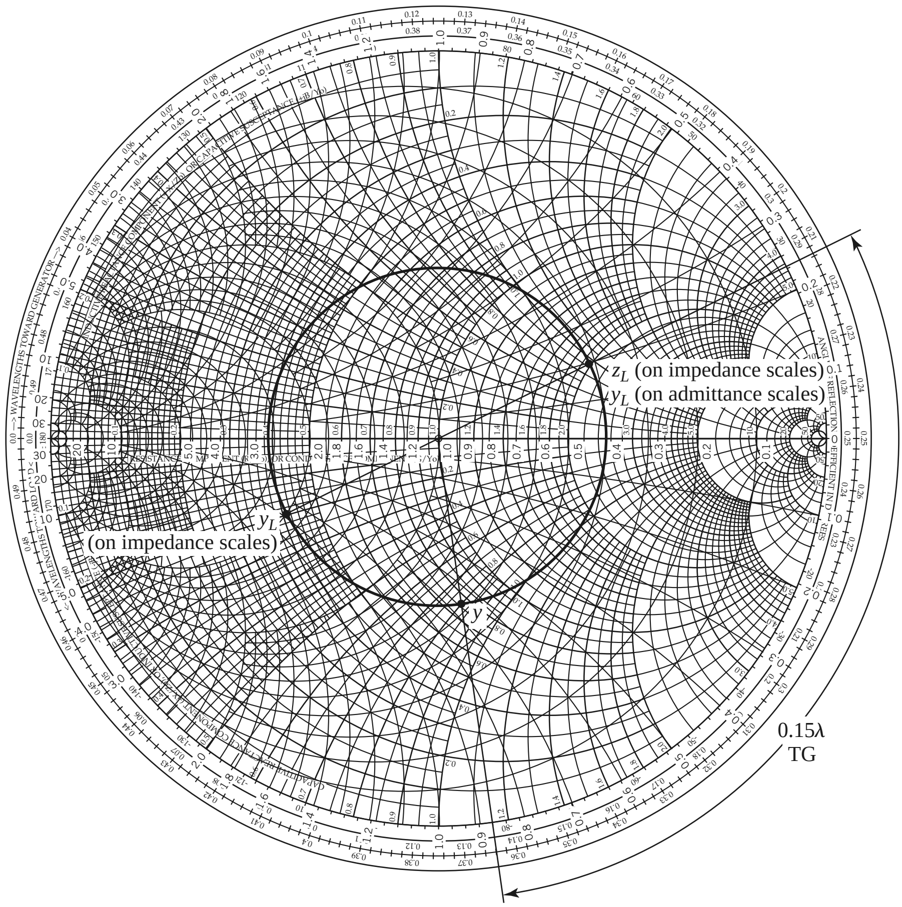


FIGURE 2.12 *ZY* Smith chart with solution for Example 2.3.

appropriate scales. Plotting z_L on the impedance scales and reading the admittance scales at this same point gives $y_L = 0.40 - j0.20$. The actual load admittance is then

$$Y_L = y_L Y_0 = \frac{y_L}{Z_0} = 0.0080 - j0.0040 \text{ S}.$$

Then, on the WTG scale, the load admittance is seen to have a reference position of 0.214λ. Moving 0.15λ past this point brings us to 0.364λ. A radial line at this point on the WTG scale intersects the SWR circle at an admittance of $y = 0.61 + j0.66$. The actual input admittance is then $Y = 0.0122 + j0.0132$ S. ■

The Slotted Line

A slotted line is a transmission line configuration (usually a waveguide or coaxial line) that allows the sampling of the electric field amplitude of a standing wave on a terminated line. With this device the SWR and the distance of the first voltage minimum from the load can be measured, and from these data the load impedance can be determined. Note that because the load impedance is, in general, a complex number (with two degrees of freedom), two distinct quantities must be measured with the slotted line to uniquely determine this impedance. A typical waveguide slotted line is shown in Figure 2.13.

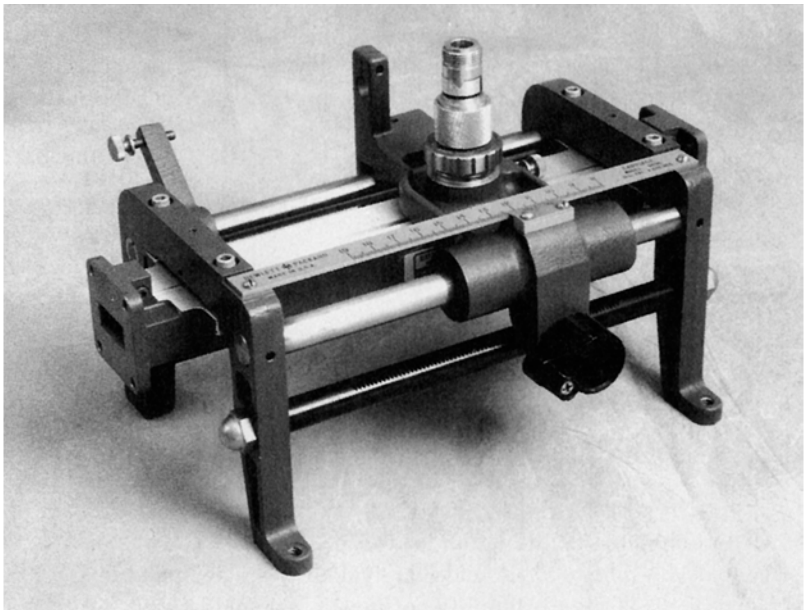


FIGURE 2.13 An X-band waveguide slotted line.

Although slotted lines used to be the principal way of measuring an unknown impedance at microwave frequencies, they have largely been superseded by the modern vector network analyzer in terms of accuracy, versatility, and convenience. The slotted line is still of some use, however, in certain applications such as high millimeter wave frequencies or where it is desired to avoid connector mismatches by connecting the unknown load directly to the slotted line, thus avoiding the use of imperfect transitions. Another reason for studying the slotted line is that it provides an unexcelled tool for learning the basic concepts of standing waves and mismatched transmission lines. We will derive expressions for finding the unknown load impedance from slotted line measurements and also show how the Smith chart can be used for the same purpose.

Assume that, for a certain terminated line, we have measured the SWR on the line and $\ell_{\min}$, the distance from the load to the first voltage minimum on the line. The load impedance Z_L can then be determined as follows. From (2.41) the magnitude of the reflection coefficient on the line is found from the standing wave ratio as

$$|\Gamma| = \frac{\mathrm{SWR} - 1}{\mathrm{SWR} + 1}. \tag{2.58}$$

From Section 2.3, we know that a voltage minimum occurs when $e^{j(\theta - 2\beta\ell)} = -1$, where θ is the phase angle of the reflection coefficient, $\Gamma = |\Gamma|e^{j\theta}$. The phase of the reflection coefficient is then

$$\theta = \pi + 2\beta\ell_{\min}, \tag{2.59}$$

where $\ell_{\min}$ is the distance from the load to the first voltage minimum. Actually, since the voltage minima repeat every $\lambda/2$, where λ is the wavelength on the line, any multiple of $\lambda/2$ can be added to $\ell_{\min}$ without changing the result in (2.59) because this just amounts to adding $2\beta n\lambda/2 = 2\pi n$ to θ, which will not change Γ. Thus, the two quantities SWR and $\ell_{\min}$ can be used to find the complex reflection coefficient Γ at the load. It is then

straightforward to use (2.43) with $\ell = 0$ to find the load impedance from Γ:

$$Z_L = Z_0 \frac{1 + \Gamma}{1 - \Gamma}. \tag{2.60}$$

The use of the Smith chart in solving this problem is best illustrated by an example.

EXAMPLE 2.4 IMPEDANCE MEASUREMENT WITH A SLOTTED LINE

The following two-step procedure has been carried out with a 50 Ω coaxial slotted line to determine an unknown load impedance:

1. A short circuit is placed at the load plane, resulting in a standing wave on the line with infinite SWR and sharply defined voltage minima, as shown in Figure 2.14a. On the arbitrarily positioned scale on the slotted line, voltage minima are recorded at

$$z = 0.2 \text{ cm}, \quad 2.2 \text{ cm}, \quad 4.2 \text{ cm}.$$

2. The short circuit is removed and replaced with the unknown load. The standing wave ratio is measured as SWR = 1.5, and voltage minima, which are not as sharply defined as those in step 1, are recorded at

$$z = 0.72 \text{ cm}, \quad 2.72 \text{ cm}, \quad 4.72 \text{ cm},$$

as shown in Figure 2.14b. Find the load impedance.

Solution

Knowing that voltage minima repeat every $\lambda/2$, we have from the data of step 1 that $\lambda = 4.0$ cm. In addition, because the reflection coefficient and input impedance also repeat every $\lambda/2$, we can consider the load terminals to be effectively located at any of the voltage minima locations listed in step 1. Thus, if we say the load is at 4.2 cm, then the data from step 2 show that the next voltage minimum away from the load occurs at 2.72 cm, giving $\ell_{\min} = 4.2 - 2.72 = 1.48 \text{ cm} = 0.37\lambda$.

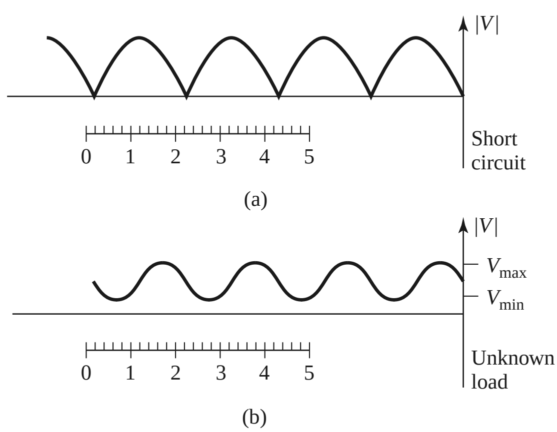


FIGURE 2.14 Voltage standing wave patterns for Example 2.4. (a) Standing wave for short-circuit load. (b) Standing wave for unknown load.

Applying (2.58)–(2.60) to these data gives

$$|\Gamma| = \frac{1.5 - 1}{1.5 + 1} = 0.2,$$

$$\theta = \pi + \frac{4\pi}{4.0}(1.48) = 86.4^\circ,$$

so

$$\Gamma = 0.2e^{j86.4^\circ} = 0.0126 + j0.1996.$$

The load impedance is then

$$Z_L = 50\left(\frac{1+\Gamma}{1-\Gamma}\right) = 47.3 + j19.7\Omega.$$

For the Smith chart version of the solution, we begin by drawing the SWR circle for SWR = 1.5, as shown in Figure 2.15; the unknown normalized load impedance must lie on this circle. The reference that we have is that the load is 0.37λ away from the first voltage minimum. On the Smith chart the position of a voltage minimum corresponds to the minimum impedance point (minimum voltage, maximum current), which is the horizontal axis (zero reactance) to the

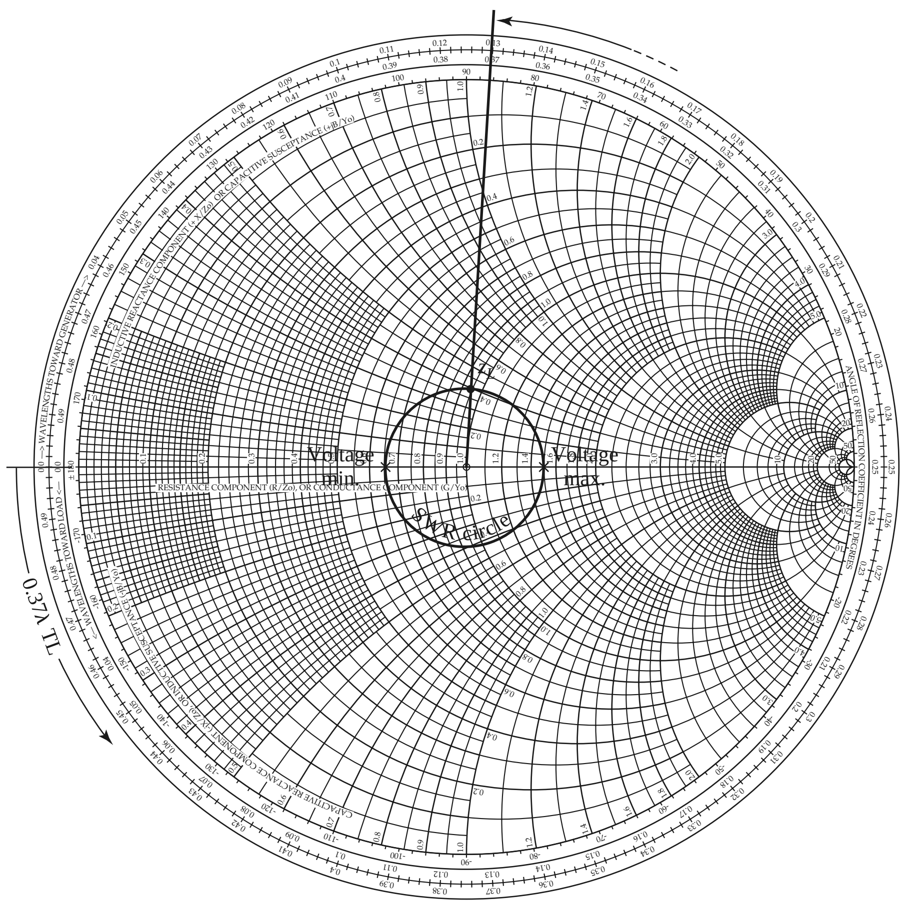


FIGURE 2.15 Smith chart for Example 2.4.

left of the origin. Thus, we begin at the voltage minimum point and move 0.37λ toward the load (counterclockwise), to the normalized load impedance point, $z_L = 0.95 + j0.4$, as shown in Figure 2.15. The actual load impedance is then $Z_L = 47.5 + j20\ \Omega$, in close agreement with the above result using equations.

Note that, in principle, voltage maxima locations could be used as well as voltage minima positions, but voltage minima are more sharply defined than voltage maxima and so usually result in greater accuracy. ■

2.5 THE QUARTER-WAVE TRANSFORMER

The quarter-wave transformer is a useful and practical circuit for impedance matching and also provides a simple transmission line circuit that further illustrates the properties of standing waves on a mismatched line. Although we will study the design and performance of quarter-wave matching transformers more extensively in Chapter 5, the main purpose here is the application of the previously developed transmission line theory to a basic transmission line circuit. We will first approach the problem from the impedance viewpoint and then show how this result can also be interpreted in terms of an infinite set of multiple reflections on the matching section.

The Impedance Viewpoint

Figure 2.16 shows a circuit employing a quarter-wave transformer. The load resistance R_L and the feedline characteristic impedance Z_0 are both real and assumed to be known. These two components are connected with a lossless piece of transmission line of (unknown) characteristic impedance Z_1 and length $\lambda/4$. It is desired to match the load to the Z_0 line by using the $\lambda/4$ section of line and so make $\Gamma = 0$ looking into the $\lambda/4$ matching section. From (2.44) the input impedance Z_{in} can be found as

$$Z_{\text{in}} = Z_1 \frac{R_L + jZ_1 \tan\beta\ell}{Z_1 + jR_L \tan\beta\ell}. \tag{2.61}$$

To evaluate this for $\beta\ell = (2\pi/\lambda)(\lambda/4) = \pi/2$, we can divide the numerator and denominator by $\tan\beta\ell$ and take the limit as $\beta\ell \to \pi/2$ to get

$$Z_{\text{in}} = \frac{Z_1^2}{R_L}. \tag{2.62}$$

In order for $\Gamma = 0$, we must have $Z_{\text{in}} = Z_0$, which yields the characteristic impedance Z_1 as

$$Z_1 = \sqrt{Z_0 R_L}, \tag{2.63}$$

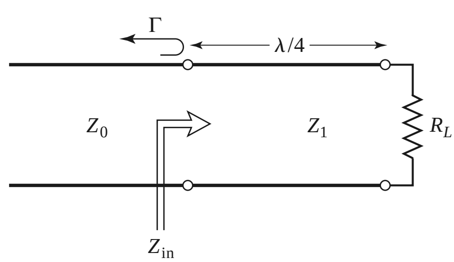


FIGURE 2.16 The quarter-wave matching transformer.

which is the geometric mean of the load and source impedances. Then there will be no standing waves on the feedline (SWR = 1), although there will be standing waves on the $\lambda/4$ matching section. In addition, the above condition applies only when the length of the matching section is $\lambda/4$ or an odd multiple of $\lambda/4$, long, so that a perfect match may be achieved at one frequency, but impedance mismatch will occur at other frequencies.

EXAMPLE 2.5 FREQUENCY RESPONSE OF A QUARTER-WAVE TRANSFORMER

Consider a load resistance $R_L = 100\ \Omega$ to be matched to a 50 Ω line with a quarter-wave transformer. Find the characteristic impedance of the matching section and plot the magnitude of the reflection coefficient versus normalized frequency, f/f_o, where f_o is the frequency at which the line is $\lambda/4$ long.

Solution
From (2.63), the necessary characteristic impedance is

$$Z_1 = \sqrt{(50)(100)} = 70.71\ \Omega.$$

The reflection coefficient magnitude is given as

$$|\Gamma| = \left| \frac{Z_{\rm in} - Z_0}{Z_{\rm in} + Z_0} \right|,$$

where the input impedance $Z_{\rm in}$ is a function of frequency as given by (2.44). The frequency dependence in (2.44) comes from the $\beta\ell$ term, which can be written in terms of f/f_o as

$$\beta\ell = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda_0}{4}\right) = \left(\frac{2\pi f}{v_p}\right)\left(\frac{v_p}{4 f_o}\right) = \frac{\pi f}{2 f_o},$$

where it is seen that $\beta\ell = \pi/2$ for $f = f_o$, as expected. For higher frequencies the matching section looks electrically longer, and for lower frequencies it looks shorter. The magnitude of the reflection coefficient is plotted versus f/f_o in Figure 2.17. ■

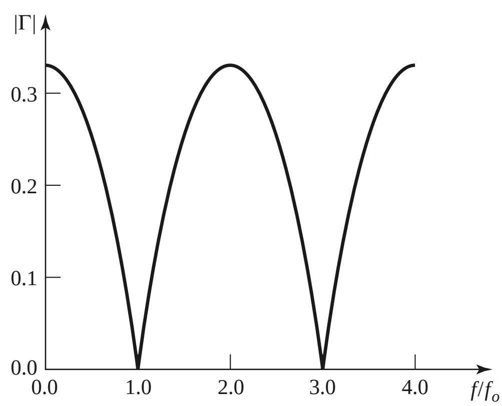


FIGURE 2.17 Reflection coefficient versus normalized frequency for the quarter-wave transformer of Example 2.5.

This method of impedance matching is limited to real load impedances, although a complex load impedance can easily be made real, at a single frequency, by transformation through an appropriate length of line.

The above analysis shows how useful the impedance concept can be when solving transmission line problems, and this method is probably the preferred method in practice. It may aid our understanding of the quarter-wave transformer (and other transmission line circuits), however, if we now look at it from the viewpoint of multiple reflections.

The Multiple-Reflectio Viewpoint

Figure 2.18 shows the quarter-wave transformer circuit with reflection and transmission coefficients defined as follows:

Γ = overall, or total, reflection coefficient of a wave incident on the $\lambda/4$ transformer (same as Γ in Example 2.5).
Γ_1 = partial reflection coefficient of a wave incident on a load Z_1, from the Z_0 line.
Γ_2 = partial reflection coefficient of a wave incident on a load Z_0, from the Z_1 line.
Γ_3 = partial reflection coefficient of a wave incident on a load R_L, from the Z_1 line.
T_1 = partial transmission coefficient of a wave from the Z_0 line into the Z_1 line.
T_2 = partial transmission coefficient of a wave from the Z_1 line into the Z_0 line.

These coefficients can be expressed as

$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \tag{2.64a}$$

$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1, \tag{2.64b}$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}, \tag{2.64c}$$

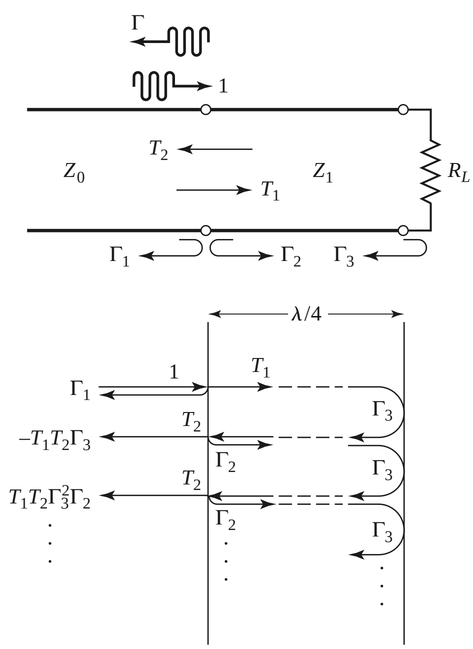


FIGURE 2.18 Multiple reflection analysis of the quarter-wave transformer.

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}, \tag{2.64d}$$

$$T_2 = \frac{2Z_0}{Z_1 + Z_0}. \tag{2.64e}$$

Now think of the quarter-wave transformer of Figure 2.18 in the time domain, and imagine a wave traveling down the Z_0 feedline toward the transformer. When the wave first hits the junction with the Z_1 line, it sees only an impedance Z_1 since it has not yet traveled to the load R_L and cannot see that effect. Part of the wave is reflected with a coefficient Γ_1, and part is transmitted onto the Z_1 line with a coefficient T_1. The transmitted wave then travels $\lambda/4$ to the load, is reflected with a coefficient Γ_3, and travels another $\lambda/4$ back to the junction with the Z_0 line. Part of this wave is transmitted through (to the left) to the Z_0 line, with coefficient T_2, and part is reflected back toward the load with coefficient Γ_2. Clearly, this process continues with an infinite number of bouncing waves, and the total reflection coefficient, Γ, is the sum of all of these partial reflections. Since each round trip path up and down the $\lambda/4$ transformer section results in a 180° phase shift, the total reflection coefficient can be expressed as

$$\begin{aligned}\Gamma &= \Gamma_1 - T_1T_2\Gamma_3 + T_1T_2\Gamma_2\Gamma_3^2 - T_1T_2\Gamma_2^2\Gamma_3^3 + \cdots \\ &= \Gamma_1 - T_1T_2\Gamma_3 \sum_{n=0}^{\infty} (-\Gamma_2\Gamma_3)^n. \end{aligned} \tag{2.65}$$

Since $|\Gamma_3| < 1$ and $|\Gamma_2| < 1$, the infinite series in (2.65) can be summed using the geometric series result that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

to give

$$\Gamma = \Gamma_1 - \frac{T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3} = \frac{\Gamma_1 + \Gamma_1\Gamma_2\Gamma_3 - T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3}. \tag{2.66}$$

The numerator of this expression can be simplified using (2.64) to give

$$\begin{aligned}\Gamma_1 - \Gamma_3\left(\Gamma_1^2 + T_1T_2\right) &= \Gamma_1 - \Gamma_3\left[\frac{(Z_1-Z_0)^2}{(Z_1+Z_0)^2} + \frac{4Z_1Z_0}{(Z_1+Z_0)^2}\right] \\ &= \Gamma_1 - \Gamma_3 = \frac{(Z_1-Z_0)(R_L+Z_1) - (R_L-Z_1)(Z_1+Z_0)}{(Z_1+Z_0)(R_L+Z_1)} \\ &= \frac{2\left(Z_1^2 - Z_0R_L\right)}{(Z_1+Z_0)(R_L+Z_1)},\end{aligned}$$

which is seen to vanish if we choose $Z_1 = \sqrt{Z_0R_L}$, as in (2.63). Then Γ of (2.66) is zero, and the line is matched. This analysis shows that the matching property of the quarter-wave transformer comes about by properly selecting the characteristic impedance and length of the matching section so that the superposition of all of the partial reflections adds to zero. Under steady-state conditions, an infinite sum of waves traveling in the same direction with the same phase velocity can be combined into a single traveling wave. Thus, the infinite set of waves traveling in the forward and reverse directions on the matching section can be reduced to two waves traveling in opposite directions. See Problem 2.25.

2.6 GENERATOR AND LOAD MISMATCHES

In Section 2.3 we treated the terminated (mismatched) transmission line assuming that the generator was matched, so that no reflections occurred at the generator. In general, however, both generator and load may present mismatched impedances to the transmission line. We will study this case and also see that the condition for maximum power transfer from the generator to the load may, in some situations, involve a standing wave on the line.

Figure 2.19 shows a transmission line circuit with arbitrary generator and load impedances Z_g and Z_ℓ, which may be complex. The transmission line is assumed to be lossless, with a length ℓ and characteristic impedance Z_0. This circuit is general enough to model most passive and active networks that occur in practice.

Because both the generator and load are mismatched, multiple reflections can occur on the line, as in the problem of the quarter-wave transformer. The present circuit could thus be analyzed using an infinite series to represent the multiple bounces, as in Section 2.5, but we will use the easier and more useful method of impedance transformation. The input impedance looking into the terminated transmission line from the generator end is, from (2.43) and (2.44),

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma_\ell e^{-2j\beta\ell}}{1 - \Gamma_\ell e^{-2j\beta\ell}} = Z_0 \frac{Z_\ell + jZ_0 \tan\beta\ell}{Z_0 + jZ_\ell \tan\beta\ell}, \tag{2.67}$$

where Γ_ℓ is the reflection coefficient of the load:

$$\Gamma_\ell = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}. \tag{2.68}$$

The voltage on the line can be written as

$$V(z) = V_o^+\left(e^{-j\beta z} + \Gamma_\ell e^{j\beta z}\right), \tag{2.69}$$

and we can find V_o^+ from the voltage at the generator end of the line, where $z = -\ell$:

$$V(-\ell) = V_g \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g} = V_o^+\left(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell}\right),$$

so that

$$V_o^+ = V_g \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g} \frac{1}{\left(e^{j\beta\ell} + \Gamma_\ell e^{-j\beta\ell}\right)}. \tag{2.70}$$

This can be rewritten, using (2.67), as

$$V_o^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j\beta\ell}}{\left(1 - \Gamma_\ell \Gamma_g e^{-2j\beta\ell}\right)}, \tag{2.71}$$

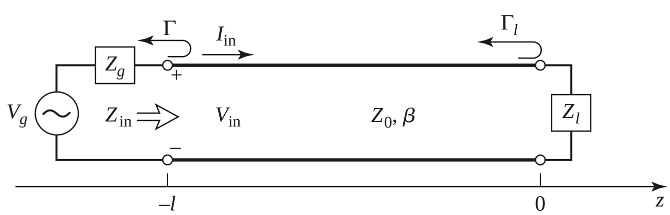


FIGURE 2.19 Transmission line circuit for mismatched load and generator.

where Γ_g is the reflection coefficient seen looking into the generator:

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}. \tag{2.72}$$

The standing wave ratio on the line is then

$$\text{SWR} = \frac{1 + |\Gamma_\ell|}{1 - |\Gamma_\ell|}. \tag{2.73}$$

The power delivered to the load is

$$P = \frac{1}{2}\text{Re}\{V_{\text{in}} I_{\text{in}}^*\} = \frac{1}{2}|V_{\text{in}}|^2 \text{Re}\left\{\frac{1}{Z_{\text{in}}}\right\} = \frac{1}{2}|V_g|^2 \left|\frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g}\right|^2 \text{Re}\left\{\frac{1}{Z_{\text{in}}}\right\}. \tag{2.74}$$

Now let $Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}}$ and $Z_g = R_g + jX_g$; then (2.74) can be reduced to

$$P = \frac{1}{2}|V_g|^2 \frac{R_{\text{in}}}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2}. \tag{2.75}$$

We now assume that the generator impedance, Z_g, is fixed, and consider three cases of load impedance.

Load Matched to Line

In this case we have $Z_l = Z_0$, so $\Gamma_\ell = 0$, and SWR = 1, from (2.68) and (2.73). Then the input impedance is $Z_{\text{in}} = Z_0$, and the power delivered to the load is, from (2.75),

$$P = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0 + R_g)^2 + X_g^2}. \tag{2.76}$$

Generator Matched to Loaded Line

In this case the load impedance Z_ℓ and/or the transmission line parameters $\beta\ell$, Z_0 are chosen to make the input impedance $Z_{\text{in}} = Z_g$, so that the generator is matched to the load presented by the terminated transmission line. Then the overall reflection coefficient, Γ, is zero:

$$\Gamma = \frac{Z_{\text{in}} - Z_g}{Z_{\text{in}} + Z_g} = 0. \tag{2.77}$$

There may, however, be a standing wave on the line since Γ_ℓ may not be zero. The power delivered to the load is

$$P = \frac{1}{2}|V_g|^2 \frac{R_g}{4\left(R_g^2 + X_g^2\right)}. \tag{2.78}$$

Observe that even though the loaded line is matched to the generator, the power delivered to the load may be less than that of (2.76), where the loaded line was not necessarily matched to the generator. Thus, we are led to the question of what is the optimum load impedance, or equivalently, what is the optimum input impedance, to achieve maximum power transfer to the load for a given generator impedance.

Conjugate Matching

Assuming that the generator series impedance Z_g is fixed, we may vary the input impedance Z_{in} until we achieve the maximum power delivered to the load. Knowing Z_{in}, it is then easy to find the corresponding load impedance Z_ℓ via an impedance transformation along

the line. To maximize P, we differentiate with respect to the real and imaginary parts of Z_{in}. Using (2.75) gives

$$\frac{\partial P}{\partial R_{\text{in}}} = 0 \rightarrow \frac{1}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2} + \frac{-2R_{\text{in}}(R_{\text{in}} + R_g)}{[(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2]^2} = 0,$$

or

$$R_g^2 - R_{\text{in}}^2 + (X_{\text{in}} + X_g)^2 = 0, \tag{2.79a}$$

and

$$\frac{\partial P}{\partial X_{\text{in}}} = 0 \rightarrow \frac{-2R_{\text{in}}(X_{\text{in}} + X_g)}{[(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2]^2} = 0,$$

or

$$X_{\text{in}}(X_{\text{in}} + X_g) = 0. \tag{2.79b}$$

Solving (2.79a) and (2.79b) simultaneously for R_{in} and X_{in} gives

$$R_{\text{in}} = R_g, \quad X_{\text{in}} = -X_g,$$

or

$$Z_{\text{in}} = Z_g^*. \tag{2.80}$$

This condition is known as *conjugate matching*, and it results in maximum power transfer to the load for a fixed generator impedance. The power delivered is, from (2.75) and (2.80),

$$P = \frac{1}{2}|V_g|^2 \frac{1}{4R_g}, \tag{2.81}$$

which is seen to be greater than or equal to the powers of (2.76) or (2.78). This is also the maximum available power from the generator. Note that the reflection coefficients Γ_ℓ, Γ_g, and Γ may be nonzero. Physically, this means that in some cases the power in the multiple reflections on a mismatched line may add in phase to deliver more power to the load than would be delivered if the line were flat (no reflections). If the generator impedance is real ($X_g = 0$), then the last two cases reduce to the same result, which is that maximum power is delivered to the load when the loaded line is matched to the generator ($R_{\text{in}} = R_g$, with $X_{\text{in}} = X_g = 0$).

Finally, note that neither matching for zero reflection ($Z_\ell = Z_0$) nor conjugate matching ($Z_{\text{in}} = Z_g^*$) necessarily yields a system with the best efficiency. For example, if $Z_g = Z_\ell = Z_0$ then both load and generator are matched (no reflections), but only half the power produced by the generator is delivered to the load (the other half is lost in Z_g), for a transmission efficiency of 50%. This efficiency can only be improved by making Z_g as small as possible.

2.7 LOSSY TRANSMISSION LINES

In practice, transmission lines have losses due to finite conductivity and/or lossy dielectric, but these losses are usually small. In many practical problems loss may be neglected, but at other times the effect of loss may be very important, as when dealing with the attenuation of a transmission line, noise introduced by a lossy line, or the Q of a resonator, for example. In this section we will study the effects of loss on transmission line behavior and show how the attenuation constant can be calculated.

The Low-Loss Line

In most practical microwave and RF transmission lines the loss is small—if this were not the case, the line would be of little practical value. When the loss is small, some approximations can be made to simplify the expressions for the general transmission line parameters of $\gamma = \alpha + j\beta$ and Z_0.

The general expression for the complex propagation constant is, from (2.5),

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{2.82}$$

which can be rearranged as

$$\begin{aligned} \gamma &= \sqrt{(j\omega L)(j\omega C)\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\ &= j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}}. \end{aligned} \tag{2.83}$$

For a low-loss line both conductor and dielectric loss will be small, and we can assume that $R \ll \omega L$ and $G \ll \omega C$. Then, $RG \ll \omega^2 LC$, and (2.83) reduces to

$$\gamma \simeq j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)}. \tag{2.84}$$

If we were to ignore the $(R/\omega L + G/\omega C)$ term we would obtain the result that γ was purely imaginary (no loss), so we will instead use the first two terms of the Taylor series expansion for $\sqrt{1+x} \simeq 1 + x/2 + \cdots$ to give the first higher order real term for γ:

$$\gamma \simeq j\omega\sqrt{LC}\left[1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right],$$

so that

$$\alpha \simeq \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right), \tag{2.85a}$$

$$\beta \simeq \omega\sqrt{LC}, \tag{2.85b}$$

where $Z_0 = \sqrt{L/C}$ is the characteristic impedance of the line in the absence of loss. Note from (2.85b) that the propagation constant β is identical to that of the lossless case of (2.12). By the same order of approximation, the characteristic impedance Z_0 can be approximated as a real quantity:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \simeq \sqrt{\frac{L}{C}}. \tag{2.86}$$

Equations (2.85)–(2.86) are known as the high-frequency, low-loss approximations for transmission lines, and they are important because they show that the propagation constant and characteristic impedance for a low-loss line can be closely approximated by considering the line as lossless.

EXAMPLE 2.6 ATTENUATION CONSTANT OF THE COAXIAL LINE

In Example 2.1 the L, C, R, and G parameters were derived for a lossy coaxial line. Assuming the loss is small, derive the attenuation constant from (2.85a) with the results from Example 2.1.

Solution
From (2.85a),

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right).$$

Using the results for R and G derived in Example 2.1 gives

$$\alpha = \frac{1}{2}\left[\frac{R_s}{\eta \ln b/a}\left(\frac{1}{a}+\frac{1}{b}\right) + \omega\epsilon''\eta\right],$$

where $\eta = \sqrt{\mu/\epsilon'}$ is the intrinsic impedance of the dielectric material filling the coaxial line. In addition, $\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon'}$ and $Z_0 = \sqrt{L/C} = (\eta/2\pi)$ $\ln b/a$. ■

This method for the calculation of attenuation requires that the line parameters L, C, R, and G be known. These can sometimes be derived using the formulas of (2.17)–(2.20), but a more direct and versatile procedure is to use the perturbation method, to be discussed shortly.

The Distortionless Line

As can be seen from the exact equations (2.82)–(2.83) for the propagation constant of a lossy line, the phase term β is generally a complicated function of frequency ω when loss is present. In particular, we note that β is generally not exactly a linear function of frequency, as in (2.85b), unless the line is lossless. If β is not a linear function of frequency (of the form $\beta = a\omega$), then the phase velocity $v_p = \omega/\beta$ will vary with frequency. The implication of this is that the various frequency components of a wideband signal will travel with different phase velocities and so arrive at the receiver end of the transmission line at slightly different times. This will lead to *dispersion*, a distortion of the signal, and is generally an undesirable effect. Granted, as we have argued, the departure of β from a linear function may be quite small, but the effect can be significant if the line is very long. This effect leads to the concept of group velocity, which we will address in detail in Section 3.10.

There is a special case, however, of a lossy line that has a linear phase factor as a function of frequency. Such a line is called a *distortionless* line, and it is characterized by line parameters that satisfy the relation

$$\frac{R}{L} = \frac{G}{C}. \tag{2.87}$$

From (2.83) the exact complex propagation constant, under the condition specified by (2.87), reduces to

$$\begin{aligned}\gamma &= j\omega\sqrt{LC}\sqrt{1 - 2j\frac{R}{\omega L} - \frac{R^2}{\omega^2 L^2}}\\ &= j\omega\sqrt{LC}\left(1 - j\frac{R}{\omega L}\right)\\ &= R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} = \alpha + j\beta,\end{aligned} \tag{2.88}$$

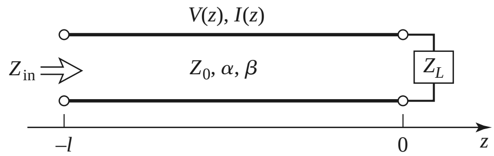


FIGURE 2.20 A lossy transmission line terminated in the impedance Z_L.

which shows that $\beta = \omega\sqrt{LC}$ is now a linear function of frequency. Equation (2.88) also shows that the attenuation constant, $\alpha = R\sqrt{C/L}$, does not depend on frequency, so that all frequency components of a signal will be attenuated by the same amount (actually, R is usually a weak function of frequency). Thus, the distortionless line is not loss free but is capable of passing a pulse or modulation envelope without distortion. To obtain a transmission line with parameters that satisfy (2.87) often requires that L be increased by adding series loading coils spaced periodically along the line.

The above theory for the distortionless line was first developed by Oliver Heaviside (1850–1925), who solved many problems in transmission line theory and reworked Maxwell's original theory of electromagnetism into the modern version that we are familiar with today [5].

The Terminated Lossy Line

Figure 2.20 shows a length ℓ of a lossy transmission line terminated in a load impedance Z_L. Thus, $\gamma = \alpha + j\beta$ is complex, but we assume the loss is small, so that Z_0 is approximately real, as in (2.86).

In (2.36), expressions for the voltage and current wave on a lossless line are given. The analogous expressions for the lossy case are

$$V(z) = V_o^+\left(e^{-\gamma z} + \Gamma e^{\gamma z}\right), \tag{2.89a}$$

$$I(z) = \frac{V_o^+}{Z_0}\left(e^{-\gamma z} - \Gamma e^{\gamma z}\right), \tag{2.89b}$$

where Γ is the reflection coefficient of the load, as given in (2.35), and V_o^+ is the incident voltage amplitude referenced at $z = 0$. From (2.42) the reflection coefficient at a distance ℓ from the load is

$$\Gamma(\ell) = \Gamma e^{-2j\beta\ell}e^{-2\alpha\ell} = \Gamma e^{-2\gamma\ell}. \tag{2.90}$$

The input impedance Z_{in} at a distance ℓ from the load is then

$$Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = Z_0\frac{Z_L + Z_0 \tanh\gamma\ell}{Z_0 + Z_L \tanh\gamma\ell}. \tag{2.91}$$

We can compute the power delivered to the input of the terminated line at $z = -\ell$ as

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2}\text{Re}\left\{V(-\ell)I^*(-\ell)\right\} = \frac{|V_o^+|^2}{2Z_0}\left(e^{2\alpha\ell} - |\Gamma|^2 e^{-2\alpha\ell}\right) \\ &= \frac{|V_o^+|^2}{2Z_0}\left(1 - |\Gamma(\ell)|^2\right)e^{2\alpha\ell}, \end{aligned} \tag{2.92}$$

where (2.89) has been used for $V(-\ell)$ and $I(-\ell)$. The power actually delivered to the load is

$$P_L = \frac{1}{2}\text{Re}\{V(0)I^*(0)\} = \frac{|V_o^+|^2}{2Z_0}(1 - |\Gamma|^2). \tag{2.93}$$

The difference in these powers corresponds to the power lost in the line:

$$P_{\text{loss}} = P_{\text{in}} - P_L = \frac{|V_o^+|^2}{2Z_0}\left[\left(e^{2\alpha\ell} - 1\right) + |\Gamma|^2\left(1 - e^{-2\alpha\ell}\right)\right]. \tag{2.94}$$

The first term in (2.94) accounts for the power loss of the incident wave, while the second term accounts for the power loss of the reflected wave; note that both terms increase as α increases.

The Perturbation Method for Calculating Attenuation

Here we derive a useful and standard technique for finding the attenuation constant of a low-loss line. The method avoids the use of the transmission line parameters L, C, R, and G and instead relies on the fields of the lossless line, with the assumption that the fields of the lossy line are not greatly different from the fields of the lossless line—hence the term, *perturbation method*.

We have seen that the power flow along a lossy transmission line, in the absence of reflections, is of the form

$$P(z) = P_o e^{-2\alpha z}, \tag{2.95}$$

where P_o is the power at the $z = 0$ plane and α is the attenuation constant we wish to determine. Now define the power loss per unit length along the line as

$$P_\ell = -\frac{\partial P}{\partial z} = 2\alpha P_o e^{-2\alpha z} = 2\alpha P(z),$$

where the negative sign on the derivative was introduced so that P_ℓ would be a positive quantity. From this, the attenuation constant can be determined as

$$\alpha = \frac{P_\ell(z)}{2P(z)} = \frac{P_\ell(z=0)}{2P_o}. \tag{2.96}$$

This equation states that α can be determined from P_o, the power on the line, and P_ℓ, the power loss per unit length of line. It is important to realize that P_ℓ can be computed from the fields of the lossless line and can account for both conductor loss [using (1.131)] and dielectric loss [using (1.92)].

EXAMPLE 2.7 USING THE PERTURBATION METHOD TO FIND THE ATTENUATION CONSTANT

Use the perturbation method to find the attenuation constant of a coaxial line having a lossy dielectric and lossy conductors.

Solution

From Example 2.1 and (2.32), the fields of the lossless coaxial line are, for $a < \rho < b$,

$$\bar{E} = \frac{V_o\hat{\rho}}{\rho \ln b/a} e^{-j\beta z},$$

$$\bar{H} = \frac{V_o\hat{\phi}}{2\pi\rho Z_0} e^{-j\beta z},$$

where $Z_0 = (\eta/2\pi)\ln b/a$ is the characteristic impedance of the coaxial line and V_o is the voltage across the line at $z = 0$. The first step is to find P_o, the power flowing on the lossless line:

$$P_o = \frac{1}{2}\text{Re}\int_S \bar{E} \times \bar{H}^* \cdot d\bar{s} = \frac{|V_o|^2}{2Z_0}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\frac{\rho d\rho d\phi}{2\pi\rho^2 \ln b/a} = \frac{|V_o|^2}{2Z_0},$$

as expected from basic circuit theory.

The loss per unit length, P_ℓ, comes from conductor loss ($P_{\ell c}$) and dielectric loss ($P_{\ell d}$). From (1.131), the conductor loss in a 1 m length of line can be found as

$$\begin{aligned} P_{\ell c} = \frac{R_s}{2}\int_S |\bar{H}_t|^2 ds &= \frac{R_s}{2}\int_{z=0}^{1}\left\{\int_{\phi=0}^{2\pi}|H_\phi(\rho = a)|^2 a\, d\phi \right. \\ &\qquad \left. + \int_{\phi=0}^{2\pi}|H_\phi(\rho = b)|^2 b\, d\phi\right\} dz \\ &= \frac{R_s|V_o|^2}{4\pi Z_0^2}\left(\frac{1}{a} + \frac{1}{b}\right). \end{aligned}$$

The dielectric loss in a 1 m length of line is, from (1.92),

$$P_{\ell d} = \frac{\omega\epsilon''}{2}\int_V |\bar{E}|^2 ds = \frac{\omega\epsilon''}{2}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}|E_\rho|^2 \rho\, d\rho\, d\phi\, dz = \frac{\pi\omega\epsilon''}{\ln b/a}|V_o|^2,$$

where ϵ'' is the imaginary part of the complex permittivity, $\epsilon = \epsilon' - j\epsilon''$. Finally, applying (2.96) gives

$$\begin{aligned} \alpha = \frac{P_{\ell c} + P_{\ell d}}{2P_o} &= \frac{R_s}{4\pi Z_0}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{\pi\omega\epsilon'' Z_0}{\ln b/a} \\ &= \frac{R_s}{2\eta \ln b/a}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{\omega\epsilon''\eta}{2}, \end{aligned}$$

where $\eta = \sqrt{\mu/\epsilon'}$. This result is seen to agree with that of Example 2.6. ■

The Wheeler Incremental Inductance Rule

Another useful technique for the practical evaluation of attenuation due to conductor loss for TEM or quasi-TEM lines is the *Wheeler incremental inductance rule* [6]. This method is based on the similarity of the equations for the inductance per unit length and resistance per unit length of a transmission line, as given by (2.17) and (2.19), respectively. In other words, the conductor loss of a line is due to current flow inside the conductor, which, as was shown in Section 1.7, is directly related to the tangential magnetic field at the surface of the conductor and thus to the inductance of the line.

From (1.131), the power loss into a cross section S of a good (but not perfect) conductor is

$$P_\ell = \frac{R_s}{2}\int_S |\bar{J}_s|^2 ds = \frac{R_s}{2}\int_S |\bar{H}_t|^2 ds \text{ W/m}^2, \tag{2.97}$$

so the power loss per unit length of a uniform transmission line is

$$P_\ell = \frac{R_s}{2}\int_C |\bar{H}_t|^2 d\ell \text{ W/m}, \tag{2.98}$$

where the line integral of (2.98) is over the cross-sectional contours of both conductors. From (2.17), the inductance per unit length of the line is

$$L = \frac{\mu}{|I|^2} \int_S |\bar{H}|^2 ds, \tag{2.99}$$

which is computed assuming the conductors are lossless. When the conductors have a small loss, the $\bar{H}$ field in the conductor is no longer zero, and this field contributes a small additional "incremental" inductance, ΔL, to that of (2.99). As discussed in Chapter 1, the fields inside the conductor decay exponentially, so that the integration into the conductor dimension can be evaluated as

$$\Delta L = \frac{\mu_0 \delta_s}{2|I|^2} \int_C |\bar{H}_t|^2 d\ell, \tag{2.100}$$

since $\int_0^\infty e^{-2z/\delta_s} dz = \delta_s/2$. (The skin depth is $\delta_s = \sqrt{2/\omega\mu\sigma}$.) Then P_ℓ from (2.98) can be written in terms of ΔL as

$$P_\ell = \frac{R_s |I|^2 \Delta L}{\mu_0 \delta_s} = \frac{|I|^2 \Delta L}{\sigma \mu_0 \delta_s^2} = \frac{|I|^2 \omega \Delta L}{2} \text{W/m}, \tag{2.101}$$

since $R_s = \sqrt{\omega\mu_0/2\sigma} = 1/\sigma\delta_s$. Then from (2.96) the attenuation due to conductor loss can be evaluated as

$$\alpha_c = \frac{P_\ell}{2P_o} = \frac{\omega \Delta L}{2Z_0}, \tag{2.102}$$

since P_o, the total power flow down the line, is $P_o = |I|^2 Z_0/2$. In (2.102), ΔL is evaluated as the change in inductance when all conductor walls recede by an amount $\delta_s/2$.

Equation (2.102) can also be written in terms of the change in characteristic impedance since

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{L}{\sqrt{LC}} = L v_p, \tag{2.103}$$

so that

$$\alpha_c = \frac{\beta \Delta Z_0}{2Z_0}, \tag{2.104}$$

where ΔZ_0 is the change in characteristic impedance when all conductor walls recede by an amount $\delta_s/2$. Yet another form of the incremental inductance rule can be obtained by using the first two terms of a Taylor series expansion for Z_0. Thus,

$$Z_0\left(\frac{\delta_s}{2}\right) \simeq Z_0 + \frac{\delta_s}{2} \frac{dZ_0}{d\ell}, \tag{2.105}$$

so that

$$\Delta Z_0 = Z_0\left(\frac{\delta_s}{2}\right) - Z_0 = \frac{\delta_s}{2} \frac{dZ_0}{d\ell},$$

where $Z_0\,(\delta_s/2)$ refers to the characteristic impedance of the line when the walls recede by $\delta_s/2$, and ℓ refers to a distance into the conductors. Then (2.104) can be written as

$$\alpha_c = \frac{\beta \delta_s}{4Z_0} \frac{dZ_0}{d\ell} = \frac{R_s}{2Z_0 \eta} \frac{dZ_0}{d\ell}, \tag{2.106}$$

where $\eta = \sqrt{\mu_0/\epsilon}$ is the intrinsic impedance of the dielectric and R_s is the surface resistivity of the conductor. Equation (2.106) is one of the most practical forms of the incremental inductance rule because the characteristic impedance is known for a wide variety of transmission lines.

EXAMPLE 2.8 USING THE WHEELER INCREMENTAL INDUCTANCE RULE TO FIND THE ATTENUATION CONSTANT

Calculate the attenuation due to conductor loss of a coaxial line using the Wheeler incremental inductance rule.

Solution
From (2.32) the characteristic impedance of the coaxial line is

$$Z_0 = \frac{\eta}{2\pi} \ln \frac{b}{a}.$$

From the incremental inductance rule of the form given in (2.106), the attenuation due to conductor loss is

$$\alpha_c = \frac{R_s}{2Z_0\eta}\frac{dZ_0}{d\ell} = \frac{R_s}{4\pi Z_0}\left(\frac{d\ln b/a}{db} - \frac{d\ln b/a}{da}\right) = \frac{R_s}{4\pi Z_0}\left(\frac{1}{b} + \frac{1}{a}\right),$$

which is seen to be in agreement with the result of Example 2.7. The negative sign on the second differentiation in this equation is because the derivative for the inner conductor is in the $-\rho$ direction (receding wall). ■

Regardless of how attenuation is calculated, measured attenuation values for practical transmission lines are usually higher. One reason for this discrepancy is the fact that realistic transmission lines have metallic surfaces with a certain amount of roughness, which increases loss, while our theoretical calculations assume perfectly smooth conductors. A quasi-empirical formula that can be used to approximately account for surface roughness for a transmission line is [7]

$$\alpha_c' = \alpha_c \left[1 + \frac{2}{\pi}\tan^{-1} 1.4 \left(\frac{\Delta}{\delta_s}\right)^2\right], \tag{2.107}$$

where α_c is the attenuation due to perfectly smooth conductors, α_c' is the attenuation corrected for surface roughness, Δ is the rms surface roughness, and δ_s is the skin depth of the conductors.

2.8 TRANSIENTS ON TRANSMISSION LINES

So far we have concentrated on the behavior of transmission lines at a single frequency, and in many cases of practical interest this viewpoint is entirely satisfactory. In some situations, however, where short pulses or very wideband signals are propagating on a transmission line, it is useful to consider wave propagation from a transient, or time domain, point of view.

In this section we will discuss the reflection of transient pulses from terminated transmission lines, including the special cases of a matched line, a short-circuited line, and an open-circuited line. We will conclude with a description of bounce diagrams, which can be used to describe multiple reflections of pulses on transmission lines.

Reflectio of Pulses from a Terminated Transmission Line

A transient transmission line circuit is shown in Figure 2.21a, where a DC source is switched on at $t = 0$. We first consider the case in which the line has a characteristic impedance of Z_0, the source impedance is Z_0, and the load impedance is Z_0. It is assumed that the voltage on the line is initially zero: $v(z, t) = 0$ for all z, for $t < 0$. We want to determine the voltage response on the transmission line as a function of time and position.

Because of the finite transit time of the line, its input impedance will appear to be equal to the characteristic impedance of the line for $t < 2\ell/v_p$, where v_p is the phase velocity of the line. In other words, the line looks infinitely long until the pulse has time to reach the load and (possibly) reflect back to the input. Therefore, when the switch closes at $t = 0$, the circuit appears as a voltage divider consisting of the source impedance and the input impedance, both being Z_0. The initial voltage on the line is thus $V_0/2$, and this voltage waveform propagates toward the load with a velocity v_p. The leading edge of the pulse will be at position z on the line at time $t = z/v_p$, as shown in Figure 2.21b.

The pulse reaches the load at time $t = \ell/v_p$. Since the load is matched to the line, there is no reflection of the pulse from the load. The circuit is now in a steady-state condition, and voltage on the line is constant: $v(z, t) = V_0/2$ for all $t > \ell/v_p$, as shown in Figure 2.21c. This is, of course, the DC value that we would expect for a voltage divider consisting of equal source and input impedances.

Next consider the transmission line circuit of Figure 2.22a, where the line is now terminated with a short circuit. Initially, the input impedance of the line again appears as Z_0, and the initial incident pulse again has an amplitude of $V_0/2$, as shown in Figure 2.22b.

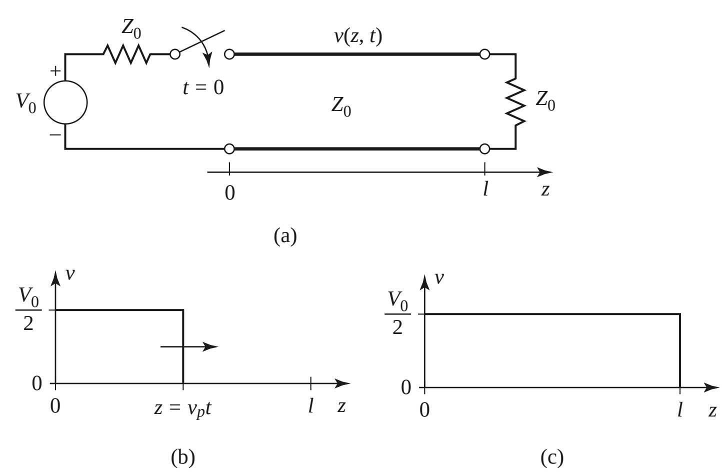


FIGURE 2.21 Transient response of a transmission line terminated with a matched load. (a) Transmission line circuit with a step function voltage source. (b) Response for $0 < t < \ell/v_p$. (c) Response for $\ell/v_p < t < 2\ell/v_p$; there is no reflection from the load.

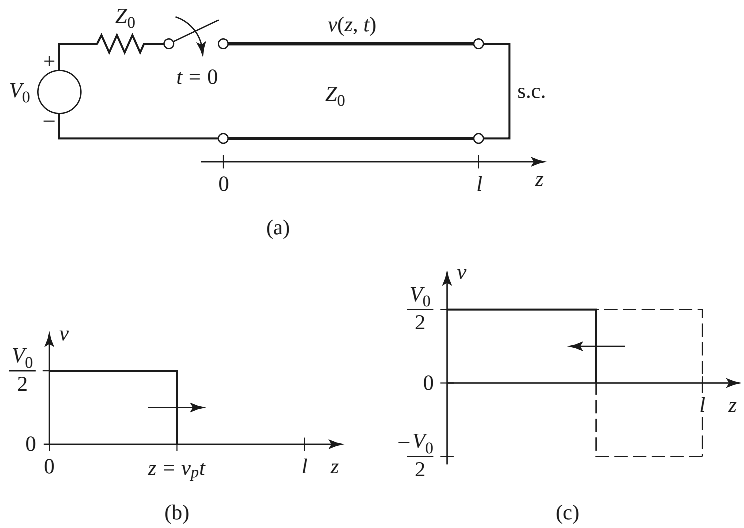


FIGURE 2.22 Transient response of a transmission line terminated with a short circuit. (a) Transmission line circuit with a step function voltage source. (b) Response for $0 < t < \ell/v_p$. (c) Response for $\ell/v_p < t < 2\ell/v_p$; the incident pulse is reflected with $\Gamma = -1$.

The short-circuit load has a reflection coefficient of $\Gamma = -1$, which has the effect of inverting the reflected pulse as it travels back toward the source. The superposition of the forward and reverse traveling pulses leads to cancellation, as shown in Figure 2.22c, for the period where $\ell/v_p < t < 2\ell/v_p$. When the return pulse reaches the source, at $t = 2\ell/v_p$, it will not be re-reflected because the source is matched to the line. The circuit is then in steady state, with zero voltage everywhere on the line. Again, this is consistent with DC circuit analysis, as the shorted line has zero electrical length at DC and thus appears as a short at its input, leading to a terminal voltage of zero. The voltage waveform at a fixed point z on the line will consist of a rectangular pulse of amplitude $V_0/2$ existing only over the time period $z/v_p < t < (2\ell - z)/v_p$. This effect can be used in practice to generate pulses of very short duration.

Finally, consider the effect of a transmission line with an open-circuit termination, as shown in Figure 2.23a. As in previous cases, the input impedance of the line initially appears as Z_0, and the initial incident pulse has an amplitude of $V_0/2$, as shown in Figure 2.23b. The open-circuit load has a reflection coefficient of $\Gamma = 1$, which reflects the incident waveform with the same polarity toward the source. The amplitudes of the forward and reverse pulses add to create a wave with an amplitude of V_0, as shown in Figure 2.23c. At $t = 2\ell/v_p$ the return pulse reaches the source, but it is not re-reflected since the source is matched to the line. The circuit is then in steady state, with a constant voltage of V_0 on the line. By DC analysis, the open-circuited line presents an open circuit at its terminals, leading to a terminal voltage equal to the source voltage.

Bounce Diagrams for Transient Propagation

The plots in Figures 2.21–2.23 show the voltage of a propagating pulse versus position along the transmission line but do not directly show the time variable, nor do they show very clearly the contribution of reflections on the waveform (especially when multiple

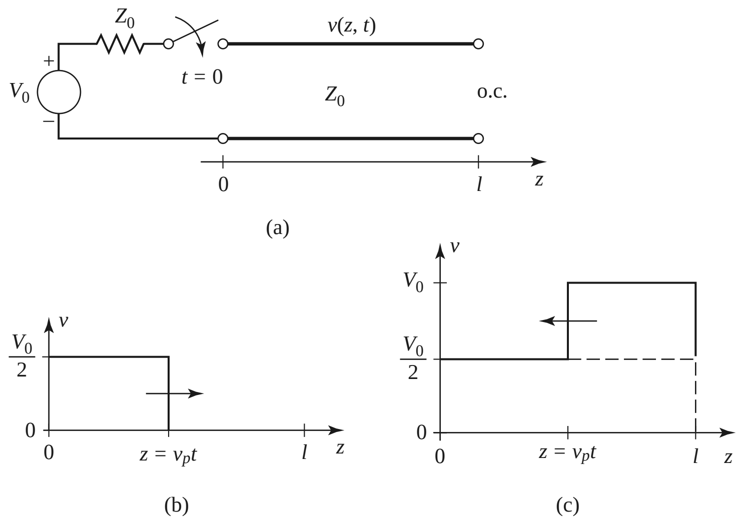


FIGURE 2.23 Transient response of a transmission line terminated with an open circuit. (a) Transmission line circuit with a step function voltage source. (b) Response for $0 < t < \ell/v_p$. (c) Response for $\ell/v_p < t < 2\ell/v_p$; the incident pulse is reflected with $\Gamma = 1$.

reflections are present). An alternative way of viewing the progress of a pulse propagating in time and position along a transmission line is with a *bounce diagram*.

As an example, Figure 2.24 shows the bounce diagram for the transient circuit of Figure 2.23a. The horizontal axis represents position on the line, while the vertical axis represents time. The ray representing the incident wave begins at $t = z = 0$ and travels to the right (increasing z) and up (for increasing t). This ray is labeled with the amplitude of the incident wave, $V_0/2$. At $t = \ell/v_p$ the incident wave reaches the open-circuit load and is reflected to produce a wave of amplitude $V_0/2$ traveling back to the source. The ray for this reflected wave thus moves to the left and up, until it reaches the source at $z = 0$ and $t = 2\ell/v_p$, at which point steady state is reached. The total voltage at any position z and time t can be easily found by drawing a vertical line through the point z and extending up from $t = 0$ to t. The total voltage is found by adding the voltages of each forward or reverse traveling wave component, as represented by the rays that intersect this vertical line.

The next example shows how a bounce diagram can be applied to circuits that have multiple reflections.

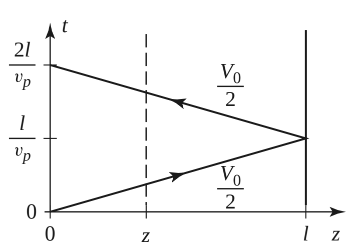


FIGURE 2.24 Bounce diagram for the transient circuit of Figure 2.23a.

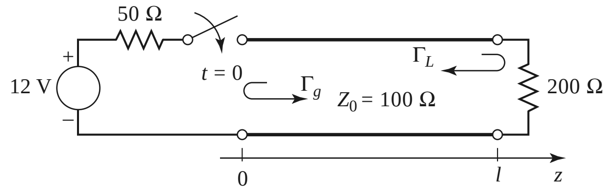
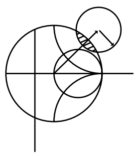


FIGURE 2.25 Circuit for Example 2.9.



EXAMPLE 2.9 BOUNCE DIAGRAM FOR A TRANSIENT CIRCUIT WITH MULTIPLE REFLECTIONS

Draw the bounce diagram for the transient circuit of Figure 2.25, including the first three reflections.

Solution

The amplitude of the incident wave is given by a voltage divider as

$$v^+ = 12\frac{100}{50 + 100} = 8.0 \text{ V}$$

The incident ray can be plotted as a line from the origin to the point $z = \ell$ and $t = \ell/v_p$. The reflection coefficients at the generator and load are

$$\Gamma_g = \frac{50 - 100}{50 + 100} = -1/3 \quad \text{and} \quad \Gamma_L = \frac{200 - 100}{200 + 100} = 1/3,$$

so the amplitude of the wave reflected from the load is 8/3 V. When this wave reaches the source, it will be reflected to form a wave of amplitude –8/9 V. The next reflection from the load will have an amplitude of –8/27 V. These four waves are shown in the bounce diagram of Figure 2.26. ■

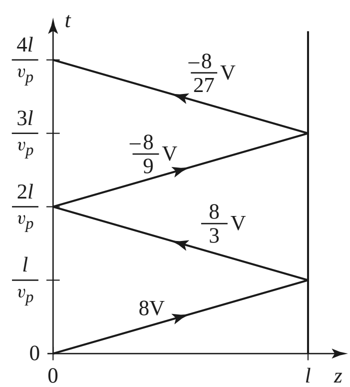


FIGURE 2.26 Bounce diagram for Example 2.9.

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PROBLEMS

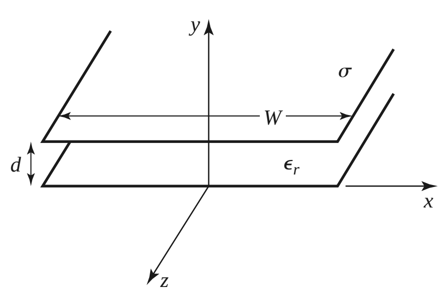
2.1 A 75 Ω coaxial line has a current $i(t, z) = 1.8 \cos(3.77 \times 10^9\, t - 18.13z)$ mA. Determine (a) the frequency, (b) the phase velocity, (c) the wavelength, (d) the relative permittivity of the line, (e) the phasor form of the current, and (f) the time domain voltage on the line.

2.2 A transmission line has the following per-unit-length parameters: $L = 0.5\ \mu$H/m, $C = 200$ pF/m, $R = 4.0\ \Omega$/m, and $G = 0.02$ S/m. Calculate the propagation constant and characteristic impedance of this line at 800 MHz. If the line is 30 cm long, what is the attenuation in dB? Recalculate these quantities in the absence of loss ($R = G = 0$).

2.3 RG-402U semirigid coaxial cable has an inner conductor diameter of 0.91 mm and a dielectric diameter (equal to the inner diameter of the outer conductor) of 3.02 mm. Both conductors are copper, and the dielectric material is Teflon. Compute the R, L, G, and C parameters of this line at 1 GHz, and use these results to find the characteristic impedance and attenuation of the line at 1 GHz. Compare your results to the manufacturer's specifications of 50 Ω and 0.43 dB/m, and discuss reasons for the difference.

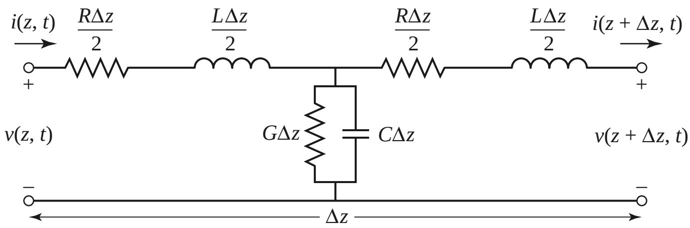
2.4 Compute and plot the attenuation of the coaxial line of Problem 2.3, in dB/m, over a frequency range of 1 MHz to 100 GHz. Use log-log graph paper.

2.5 For the parallel plate line shown in the accompanying figure, derive the R, L, G, and C parameters. Assume $W \gg d$.

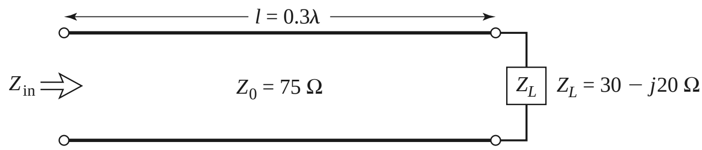


2.6 For the parallel plate line of Problem 2.5, derive the telegrapher equations using the field theory approach.

2.7 Show that the T-model of a transmission line shown in the accompanying figure also yields the telegrapher equations derived in Section 2.1.



2.8 A lossless transmission line of electrical length $\ell = 0.3\lambda$ is terminated with a complex load impedance as shown in the accompanying figure. Find the reflection coefficient at the load, the SWR on the line, the reflection coefficient at the input of the line, and the input impedance to the line.



2.9 A 75 Ω coaxial transmission line has a length of 2.0 cm and is terminated with a load impedance of $37.5 + j75\ \Omega$. If the relative permittivity of the line is 2.56 and the frequency is 3.0 GHz, find the input impedance to the line, the reflection coefficient at the load, the reflection coefficient at the input, and the SWR on the line.

2.10 A terminated transmission line with $Z_0 = 60\ \Omega$ has a reflection coefficient at the load of $\Gamma = 0.4\angle 60°$. (a) What is the load impedance? (b) What is the reflection coefficient 0.3λ away from the load? (c) What is the input impedance at this point?

2.11 A 100 Ω transmission line has an effective dielectric constant of 1.65. Find the shortest open-circuited length of this line that appears at its input as a capacitor of 5 pF at 2.5 GHz. Repeat for an inductance of 5 nH.

2.12 A lossless transmission line is terminated with a 100 Ω load. If the SWR on the line is 1.5, find the two possible values for the characteristic impedance of the line.

2.13 Let Z_{sc} be the input impedance of a length of coaxial line when one end is short-circuited, and let Z_{oc} be the input impedance of the line when one end is open-circuited. Derive an expression for the characteristic impedance of the cable in terms of Z_{sc} and Z_{oc}.

2.14 A radio transmitter is connected to an antenna having an impedance $80 + j40\ \Omega$ with a 50 Ω coaxial cable. If the 50 Ω transmitter can deliver 30 W when connected to a 50 Ω load, how much power is delivered to the antenna?

2.15 Calculate standing wave ratio, reflection coefficient magnitude, and return loss values to complete the entries in the following table:

SWR	$\|\Gamma\|$	RL (dB)
1.00	0.00	∞
1.01	—	—
—	0.01	—
1.05	—	—
—	—	30.0
1.10	—	—
1.20	—	—
—	0.10	—
1.50	—	—
—	—	10.0
2.00	—	—
2.50	—	—

2.16 The transmission line circuit in the accompanying figure has $V_g = 15$ V rms, $Z_g = 75\ \Omega$, $Z_0 = 75\ \Omega$, $Z_L = 60 - j40\ \Omega$, and $\ell = 0.7\lambda$. Compute the power delivered to the load using three different techniques:

(a) Find Γ and compute

$$P_L = \left(\frac{V_g}{2}\right)^2 \frac{1}{Z_0}(1 - |\Gamma|^2);$$

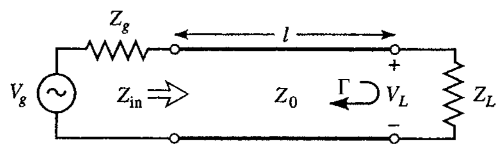
(b) find Z_{in} and compute

$$P_L = \left|\frac{V_g}{Z_g + Z_{\text{in}}}\right|^2 \text{Re}\{Z_{\text{in}}\};$$

(c) find V_L and compute

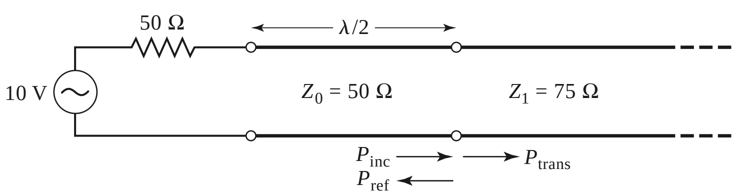
$$P_L = \left|\frac{V_L}{Z_L}\right|^2 \text{Re}\{Z_L\}.$$

Discuss the rationale for each of these methods. Which of these methods can be used if the line is not lossless?

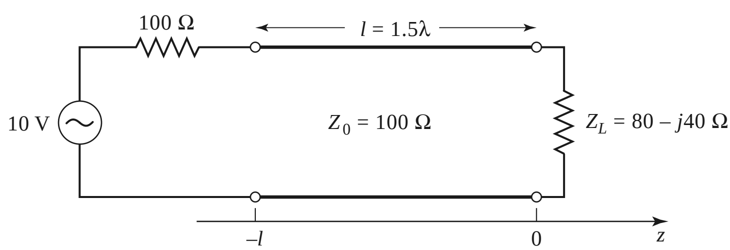


2.17 For a purely reactive load impedance of the form $Z_L = jX$, show that the reflection coefficient magnitude $|\Gamma|$ is always unity. Assume that the characteristic impedance Z_0 is real.

2.18 Consider the transmission line circuit shown in the accompanying figure. Compute the incident power, the reflected power, and the power transmitted into the infinite 75 Ω line. Show that power conservation is satisfied.



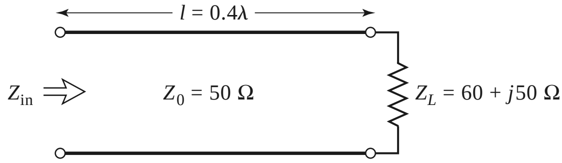
2.19 A generator is connected to a transmission line as shown in the accompanying figure. Find the voltage as a function of z along the transmission line. Plot the magnitude of this voltage for $-\ell \le z \le 0$.



2.20 Use the Smith chart to find the following quantities for the transmission line circuit shown in the accompanying figure:

(a) The SWR on the line.
(b) The reflection coefficient at the load.
(c) The load admittance.
(d) The input impedance of the line.
(e) The distance from the load to the first voltage minimum.

(f) The distance from the load to the first voltage maximum.



2.21 Use the Smith chart to find the shortest lengths of a short-circuited 75 Ω line to give the following input impedance:

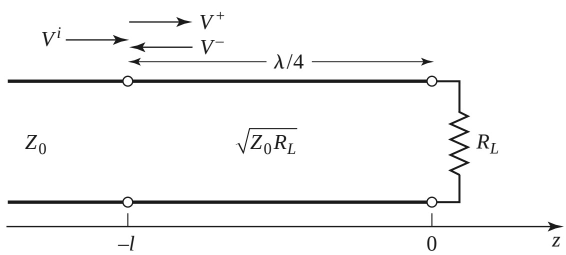
(a) $Z_{\text{in}} = 0$.
(b) $Z_{\text{in}} = \infty$.
(c) $Z_{\text{in}} = j75\ \Omega$.
(d) $Z_{\text{in}} = -j50\ \Omega$.
(e) $Z_{\text{in}} = j10\ \Omega$.

2.22 Repeat Problem 2.21 for an open-circuited length of 75 Ω line.

2.23 A slotted-line experiment is performed with the following results: distance between successive minima = 2.1 cm; distance of first voltage minimum from load = 0.9 cm; SWR of load = 2.5. If $Z_0 = 50\ \Omega$, find the load impedance.

2.24 Design a quarter-wave matching transformer to match a 40 Ω load to a 75 Ω line. Plot the SWR for $0.5 \le f/f_o \le 2.0$, where f_o is the frequency at which the line is λ/4 long.

2.25 Consider the quarter-wave matching transformer circuit shown in the accompanying figure. Derive expressions for V^+ and V^-, the respective amplitudes of the forward and reverse traveling waves on the quarter-wave line section, in terms of V^i, the incident voltage amplitude.



2.26 Derive equation (2.71) from (2.70).

2.27 In Example 2.7, the attenuation of a coaxial line due to finite conductivity is

$$\alpha_c = \frac{R_s}{2\eta \ln b/a}\left(\frac{1}{a} + \frac{1}{b}\right).$$

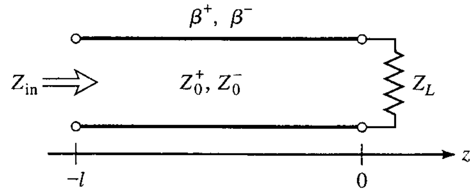
Show that α_c is minimized for conductor radii such that $x \ln x = 1 + x$, where $x = b/a$. Solve this equation for x, and show that the corresponding characteristic impedance for $\epsilon_r = 1$ is 77 Ω.

2.28 Compute and plot the factor by which attenuation is increased due to surface roughness, for rms roughness ranging from 0 to 0.01 mm. Assume copper conductors at 10 GHz.

2.29 A 50 Ω transmission line is matched to a 10 V source and feeds a load $Z_L = 100\ \Omega$. If the line is 2.3λ long and has an attenuation constant $\alpha = 0.5$ dB/λ, find the powers that are delivered by the source, lost in the line, and delivered to the load.

2.30 Consider a nonreciprocal transmission line having different propagation constants, β^+ and β^-, for propagation in the forward and reverse directions, with corresponding characteristic impedances Z_0^+ and Z_0^-. (An example of such a line could be a microstrip transmission line on a magnetized ferrite

substrate.) If the line is terminated as shown in the accompanying figure, derive expressions for the reflection coefficient and impedance seen at the input of the line.



2.31 Plot the bounce diagram for the transient circuit shown in the accompanying figure. Include at least three reflections. What is the total voltage at the midpoint of the line ($z = l/2$), at time $t = 3\ell/v_p$?